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The list chromatic index of simple graphs whose odd cycles intersect in at most one edge

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ABSTRACT

We study the class of simple graphs $\mathcal{G}^*$ for which every pair of distinct odd cycles intersect in at most one edge. We give a structural characterization of the graphs in $\mathcal{G}^*$ and prove that every $G \in \mathcal{G}^*$ satisfies the list-edge-coloring conjecture. When $\Delta(G) \geq 4$, we in fact prove a stronger result about kernel-perfect orientations in $L(G)$ which implies that G is $(m\Delta(G) : m)$-edge-choosable and $\Delta(G)$-edge-paintable for every $m \geq 1$.

1. Introduction

In this paper all graphs are assumed to be simple unless otherwise indicated.

A fundamental characterization of bipartite graphs, proved by König [7], is that a graph is bipartite if and only if it contains no odd cycle. Hsu, Ikura, and Nemhauser [5], and independently Maffray [8], generalized this result, giving the following structural characterization of the class $\mathcal{G}_1$ of graphs containing no odd cycles of length longer than 3. Here, a *block* of a graph is a maximal connected subgraph of G having no cut vertex, and the *join* $G \vee H$ of two graphs G and H is the graph obtained from their disjoint union by adding all edges between vertices of G and vertices of H.

Theorem 1.1 (*Hsu–Ikura–Nemhauser [5], Maffray [8]*)**.** *A graph G lies in $\mathcal{G}_1$ if and only each of its blocks B satisfies one of the following conditions:*

- *B is bipartite, or*
- *$B \cong K_4$, or*
- *$B \cong K_2 \vee \overline{K_r}$ for some $r \geq 1$.*

In this paper we study graphs where some longer odd cycles are allowed. Let $\mathcal{G}^*$ be the class of graphs G in which odd cycles intersect in at most one edge, i.e., for any distinct odd cycles C_1, C_2 in G, we have $|E(C_1) \cap E(C_2)| \leq 1$. Since any two distinct triangles in a graph intersect in at most one edge, we immediately have that $\mathcal{G}_1 \subseteq \mathcal{G}^*$. Building on Theorem 1.1, our first result is the following structural characterization of the graphs in $\mathcal{G}^*$. Here, for positive integers $p_1, \ldots, p_k$, the Θ-graph $\Theta_{p_1,\ldots,p_k}$ is the graph obtained from a pair of vertices $\{x_1, x_2\}$ joined by k internally disjoint paths, with the ith path containing p_i edges. (In particular, if some $p_i = 1$ then the corresponding path is just an edge joining x_1 and x_2.) Fig. 1 shows $\Theta_{1,2,4}$.

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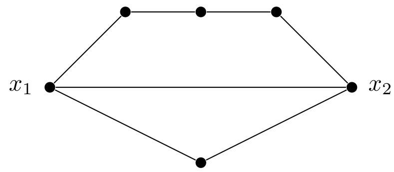

Fig. 1. The graph $\Theta_{1,2,4}$.

Theorem 1.2. *A graph G lies in $\mathcal{G}^*$ if and only if each of its blocks B satisfies one of the following conditions:*

- *B is bipartite, or*
- *$B \cong K_4$, or*
- *$B \cong \Theta_{1,p_1,\ldots,p_r}$, where $p_1, \ldots, p_r$ are even.*

Since $K_2 \vee \overline{K_r} \cong \Theta_{1,2,\ldots,2}$, with the 2 repeated r times, we see that the blocks permitted in Theorem 1.2 generalize the blocks permitted in Theorem 1.1, as required by the inclusion $\mathcal{G}_1 \subseteq \mathcal{G}^*$. We prove this characterization in Section 2.

We use our structural characterization of the graphs in $\mathcal{G}^*$ to prove a result about the list chromatic index of these graphs. A *list assignment* to the edges of a graph G is a function ℓ that assigns each edge $e \in E(G)$ a list of colors $\ell(v)$. (Typically one uses the letter L to denote a list assignment; we use ℓ to avoid conflict with the notation $L(G)$ for line graphs.) An *ℓ-edge-coloring* of G is a function ϕ defined on $E(G)$ such that $\phi(e) \in \ell(e)$ for all e and such that $\phi(e_1) \neq \phi(e_2)$ whenever e_1, e_2 are adjacent. A *k-edge-coloring* of G is an ℓ-edge-coloring for the list assignment with $\ell(e) = \{1, \ldots, k\}$ for all $e \in E(G)$; the *chromatic index* of G, written $\chi'(G)$, is the smallest nonnegative k such that G admits a k-edge-coloring. If $f : E(G) \to \mathbb{N}$ and G has a proper ℓ-edge-coloring whenever $|\ell(e)| \geq f(e)$ for all e, we say that G is *f-edge-choosable*. In particular, if G is f-edge-choosable when $f(e) = k$ for all e, we say that G is *k-edge-choosable*. The *list chromatic index* of G, written $\chi'_l(G)$, is the smallest nonnegative k such that G is k-edge-choosable.

It is clear that $\chi'_l(G) \geq \chi'(G)$ for every graph G. The list-edge-coloring conjecture (attributed to many sources, some as early as 1975; see [6]) asserts that equality always holds. In a breakthrough result in 1995, Galvin [4] proved the conjecture for bipartite graphs. Peterson and Woodall [9,10] later extended this to the class $\mathcal{G}_1$. Here we prove, in Section 3, that the list-edge-coloring conjecture holds for $\mathcal{G}^*$.

Theorem 1.3. *If $G \in \mathcal{G}^*$, then $\chi'_l(G) = \chi'(G)$.*

As $\mathcal{G}_1 \subseteq \mathcal{G}^*$, Theorem 1.3 extends the work of the above-mentioned authors, in particular by allowing odd cycles of any length. However, while our proof is limited to simple graphs, Galvin's proof also holds for bipartite multigraphs. Peterson and Woodall's proof works for multigraphs too: they proved that $\chi'_l(G) = \chi'(G)$ for all multigraphs G whose underlying simple graph is in $\mathcal{G}_1$. It would be desirable to extend Theorem 1.3 to all multigraphs G whose underlying simple graph is in $\mathcal{G}^*$, but we have not been able to do so.

In looking at Theorem 1.3, recall that Vizing's Theorem [12] says that every graph G has chromatic index $\Delta(G)$ or $\Delta(G)+1$. We shall see that every connected graph in $\mathcal{G}^*$, aside from odd cycles, satisfies $\chi'_l(G) = \chi'(G) = \Delta(G)$. When $\Delta(G) \geq 4$, we actually prove something stronger than the fact that $G \in \mathcal{G}^*$ is $\Delta(G)$-edge-choosable: we prove a result about orienting $L(G)$ which implies results for two generalized notions of choosability. Before describing these, we first discuss the connection between coloring and kernels in digraphs.

An *orientation* of a graph G is any digraph obtained by replacing each edge $uv \in E(G)$ with the arc (u, v), the arc (v, u), or both of these arcs. A *kernel* in a digraph D is an independent set of vertices S such that every vertex in $D - S$ has an out-neighbor in S. A digraph D is said to be *kernel-perfect* if every induced subdigraph of D, including D itself, has a kernel. Coloring and kernels are linked by the following lemma of Bondy, Boppana and Siegel (see [4]); here we state the lemma for line graphs (edge-coloring) only.

Lemma 1.4 (*Bondy–Boppana–Siegel*)**.** *If D is a kernel-perfect orientation of a line graph $L(G)$ and $f(e) \geq 1 + d^+_D(e)$ for all $e \in E(G)$, then G is f-edge-choosable.*

Remark. A word of caution is in order regarding our use of the word "orientation". In an orientation of a line graph $L(G)$, we explicitly allow the possibility that both of the arcs (e, f) and (f, e) are present, even when e and f are not parallel edges. This possibility is also allowed by Maffray [8]. However, some papers in the literature implicitly *forbid* such 2-cycles, such as the paper of Borodin, Kostochka, and Woodall [1] generalizing results of [8]. We mention the distinction here in the hope of avoiding future confusion about which notion of "orientation" is used in this paper.

Let H be a graph, and let $f : V(H) \to \mathbb{N}$. We say an orientation D of H is *f-kernel-perfect* if it is kernel-perfect and $f(v) \geq 1 + d^+_D(v)$ for all $v \in V(H)$; if $f(v) = k$ for all v then we say *k-kernel-perfect*. We say a graph G is *f-edge-orientable* if $L(G)$ admits an f-kernel-perfect orientation. Using this terminology, the above lemma says that f-edge-orientability implies f-edge-choosability. In Section 3, we prove the following result: