Discrete Mathematics 341 (2018) 723–731

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

A sharp lower bound on Steiner Wiener index for trees with given diameter

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ARTICLE INFO

Article history:
Received 8 June 2017
Received in revised form 22 October 2017
Accepted 13 November 2017

Keywords:
Distance
Steiner distance
Steiner Wiener index
Caterpillar trees

ABSTRACT

Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. For a subset S of $V(G)$, the *Steiner distance* $d(S)$ of S is the minimum size of a connected subgraph whose vertex set contains S. For an integer k with $2 \le k \le n-1$, the *Steiner k-Wiener index* $\mathrm{SW}_k(G)$ is $\sum_{S\subseteq V(G),|S|=k} d(S)$. In this paper, we introduce some transformations for trees that do not increase their Steiner k-Wiener index for $2 \le k \le n-1$. Using these transformations, we get a sharp lower bound on Steiner k-Wiener index for trees with given diameter, and obtain the corresponding extremal graph as well.

1. Introduction

Throughout this paper all graphs are connected and simple, and all notations and terminologies not described here are standard in [1]. For a graph G and two vertices $u, v \in V(G)$, the *distance* between u and v, denoted by $d_G(u, v)$, is the length of a shortest path connecting u and v. The *diameter* $d(G)$ of G is the largest distance between any two vertices. The *Wiener index* $W(G)$ of a graph G is the sum of distances between each pair of vertices, that is,

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u, v).$$

The Wiener index is an important distance-based graph invariant. It was proposed by Harold Wiener [11] in 1947. He found that there exist correlations between the boiling points of paraffins and their molecular structure. The study of the Wiener index in mathematics dates back to the 1970s [4]. Since then, the Wiener index obtained wide attention and many splendid results have been obtained, see the surveys [3,6,7,12].

Let G be a graph with vertex set V and edge set E. For a subset S of V, the *Steiner distance* $d_G(S)$ of S is the minimum size of a connected subgraph whose vertex set contains S, that is,

$$d_G(S) = \min\{|E(H)| : H \text{ is a connected subgraph of } G \text{ with } S \subseteq V(H)\}.$$

This concept was proposed by Chartrand et al. [2] in 1989. Note that the size of the spanning tree of H is not greater than $|E(H)|$. Therefore, the Steiner distance can be written as

$$d_G(S) = \min\{|E(T)| : T \text{ is a subtree of } G \text{ with } S \subseteq V(T)\}.$$

Taking $S = \{u, v\}$, we see that $d_G(S) = d_G(u, v)$. Thus the concept of Steiner distance is a natural generalization of the concept of classical distance.

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https://doi.org/10.1016/j.disc.2017.11.009
0012-365X/

With respect to the concept of Steiner distance, Li et al. [8] generalized the concept of Wiener index by Steiner Wiener index. For an integer k with $2 \le k \le n-1$, the *Steiner k-Wiener index* $\mathrm{SW}_k(G)$ of G is the sum of Steiner k-distances of all subsets S of V with $|S| = k$, that is,

$$\mathrm{SW}_k(G) = \sum_{S \subseteq V, |S|=k} d_G(S).$$

The classical Wiener index is just the special case of Steiner k-Wiener index for $k = 2$. The application of Steiner Wiener index was introduced in [5]. Recently, Mao et al. [10] established expressions for the Steiner k-Wiener index on the join, corona, cluster, lexicographical product, and Cartesian product of graphs.

In 1976, Entringer et al. [4] obtained the lower and upper bounds on Wiener index for trees, that is,

$$(n-1)^2 \le W(T) \le \binom{n+1}{3}$$

and the star S_n minimizes the Wiener index and the path P_n maximizes the Wiener index. Recently, Li et al. [8] generalized this result to the Steiner Wiener index, that is,

$$\binom{n-1}{k-1}(n-1) \le \mathrm{SW}_k(T) \le (k-1)\binom{n+1}{k+1}$$

for $2 \le k \le n-1$, and the star S_n and the path P_n attain the lower and upper bounds, respectively. In 2008, Liu et al. [9] characterized the tree with smallest Wiener index among all trees with given diameter. Naturally, we would like to generalize this result to the Steiner Wiener index. In Section 2, we introduce some transformations for a tree which do not increase its Steiner Wiener index. In Section 3, we give a sharp lower bound on the Steiner Wiener index for trees with given diameter, and obtain the corresponding extremal graph as well.

2. Transformations for trees

Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, denote by $d(v)$ and $N(v)$ the degree and the neighbourhood of v, respectively. As usual, we write P_n, C_n and $K_{a,b}$ for the path, the cycle and the complete bipartite graphs, respectively. For two integers n and d with $2 \le d \le n-1$, let $\mathcal{T}(n)$ be the family of trees on n vertices and $\mathcal{T}(n,d) = \{T \in \mathcal{T}(n) : d(T) = d\}$. Clearly, $\mathcal{T}(2) = \{P_2\}$, $\mathcal{T}(3) = \{P_3\}$, $\mathcal{T}(n,2) = \{K_{1,n-1}\}$ and $\mathcal{T}(n,n-1) = \{P_n\}$. Each of them contains only one graph whose Steiner Wiener index is clear. Therefore, we only consider $\mathcal{T}(n,d)$ with $n \ge 4$ and $3 \le d \le n-2$. In this part, we will introduce some transformations for a tree, which do not increase its Steiner Wiener index.

We start with a useful combinatorial inequality.

Lemma 2.1. *Let a, b and k be three positive integers such that $a \le b$. If $2 \le k \le b+1$, then $\binom{a}{k} + \binom{b}{k} < \binom{a-1}{k} + \binom{b+1}{k}$; if $k \ge b+2$, then $\binom{a}{k} + \binom{b}{k} = \binom{a-1}{k} + \binom{b+1}{k}$.*

Proof. Note that $\binom{n-1}{m-1} + \binom{n-1}{m} = \binom{n}{m}$ for two positive integers m and n. It follows that

$$\left[\binom{a-1}{k} + \binom{b+1}{k}\right] - \left[\binom{a}{k} + \binom{b}{k}\right] = \left[\binom{b+1}{k} - \binom{b}{k}\right] - \left[\binom{a}{k} - \binom{a-1}{k}\right] = \binom{b}{k-1} - \binom{a-1}{k-1}.$$

If $2 \le k \le b+1$ then $\binom{b}{k-1} - \binom{a-1}{k-1} > 0$; if $k \ge b+2$ then $\binom{b}{k-1} - \binom{a-1}{k-1} = 0 - 0 = 0$. This completes the proof. $\square$

Let $T \in \mathcal{T}(n)$ and $e \in E(T)$ such that $e = v_1 v_2$. We say that v_1 and v_2 are the *left end* and the *right end* of e, respectively. Denote by

$$N_l^{(T)}(e) = \{v \in V(T) : d(v, v_1) < d(v, v_2)\}, N_r^{(T)}(e) = \{v \in V(T) : d(v, v_1) > d(v, v_2)\}$$

and $n_l^{(T)}(e) = |N_l^{(T)}(e)|$, $n_r^{(T)}(e) = |N_r^{(T)}(e)|$. In other words, $N_l^{(T)}(e)$ and $N_r^{(T)}(e)$ are the vertex sets of the components of $G - e$ containing v_1 and v_2, respectively. By the definitions, $V(T) = N_l^{(T)}(e) \cup N_r^{(T)}(e)$ and $n = n_l^{(T)}(e) + n_r^{(T)}(e)$. Denote by $\gamma^{(T)}(e) = \min\{n_l^{(T)}(e), n_r^{(T)}(e)\}$ and $\eta^{(T)}(e) = \max\{n_l^{(T)}(e), n_r^{(T)}(e)\}$. Obviously, $n_l^{(T)}(v_1v_2)$ and $n_r^{(T)}(v_1v_2)$ depend on the order of v_1 and v_2, but $\gamma^{(T)}(v_1v_2)$ and $\eta^{(T)}(v_1v_2)$ do not. When the tree T is clear from the context, we delete T from the notations like $N_l^{(T)}(e)$, $n_l^{(T)}(e)$ and $\gamma^{(T)}(e)$. Li et al. give a useful formula to calculate the Steiner Wiener index of a tree.

Lemma 2.2 (*Theorem 4.3 of [8]*)**.** *Let k be an integer such that $2 \le k \le n$. If T is a tree, then for its Steiner k-Wiener index holds*

$$SW_k(T) = \sum_{e \in E(T)} \sum_{i=1}^{k-1} \binom{n_l(e)}{i}\binom{n_r(e)}{k-i}.$$

Note that $\mathrm{SW}_n(T) = n-1$ for all trees on n vertices. We only consider $\mathrm{SW}_k(T)$ for $2 \le k \le n-1$. Since $\{\gamma(e), \eta(e)\} = \{n_l(e), n_r(e)\}$, the formula given in Lemma 2.2 can be simplified as follows.