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Fan-type condition on disjoint cycles in a graph[☆]

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ABSTRACT

Let k be a positive integer. In this paper, we prove that for a graph G with at least $4k$ vertices, if $\max\{d(x), d(y)\} \geq 2k$ for any pair of nonadjacent vertices $\{x, y\} \subseteq V(G)$, then G contains k disjoint cycles. This generalizes the results given by Corrádi and Hajnal (1963), Enomoto (1998), and Wang (1999).

1. Introduction

In this paper, we consider only finite undirected graphs without loops and multiple edges. Let $G = (V, E)$ be a graph, the order of G is $|G| = |V|$. For a vertex $u \in V(G)$ and a subgraph H of G, $N(u, H)$ is the set of neighbours of u contained in H and let $d_H(u) = |N(u, H)|$. Clearly, $d_G(u)$ is the degree of u in G and we write $d(u)$ to replace $d_G(u)$. The distance between two vertices u and v, denoted by $d(u, v)$, is the minimum length of a $u - v$ path. The minimum degree of G is denoted by $\delta(G)$ and we define $\sigma_2(G) = \min\{d(x) + d(y) | x \in V, y \in V, xy \notin E(G)\}$. For a subset $U \subseteq V(G)$, $G[U]$ denotes the subgraph of G induced by U. A set of subgraphs of G is said to be disjoint if no two of them have any common vertex in G.

Disjoint cycles in graphs have been considered since 1960s. Corrádi and Hajnal [3] proved if G is a graph of order at least $3k$ with minimum degree at least $2k$, then G contains k disjoint cycles. [8] gave a result that generalizes $\delta(G)$ to $\sigma_2(G)$. Unfortunately, no proof of the result was given. In 1998 and 1999, Enomoto [5] and Wang [11] proved the following result, independently.

Theorem 1.1 ([5,11]). *Let k be a positive integer. If $|G| \geq 3k$ and $\sigma_2(G) \geq 4k - 1$, then G contains k disjoint cycles.*

As observed in [5,11], the following example shows that the degree condition given in Theorem 1.1 is sharp. Let E_n be the graph of order $n \geq k + 1$ with no edges. The graph $K_{2k-1} + E_n$ is a graph of order $(2k + n - 1)$ obtained from K_{2k-1} and E_n by joining every vertex of K_{2k-1} to every vertex of E_n. This graph does not contain k disjoint cycles. But $d(x) + d(y) = 4k - 2$ for every pair of nonadjacent vertices x and y.

Very recently, Jiao, Wang and Yan improved the result of Corrádi and Hajnal in a new direction.

Theorem 1.2 ([7]). *Let G be a connected graph of order at least $3k$, where k is a positive integer. Suppose that $d(x) + d(y) \geq 4k$ for every pair of vertices x and y of distance 2 in G. Then, with one exception, G contains k disjoint cycles.*

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Let G be graph of order n. In [6], Fan-condition is given as follows: If $d(x, y) = 2$, then $\max\{d(x), d(y)\} \geq \frac{n}{2}$ for any two vertices x and y of G.

For a graph G with at least $4k$ vertices, this paper improves the result of Corrádi and Hajnal by considering Fan-type condition "$\max\{d(x), d(y)\} \geq 2k$ for any two vertices x and y of G with $xy \notin E$". The following theorem is proved.

Theorem 1.3. *Let k be a positive integer and let G be a simple graph with at least $4k$ vertices. If $\max\{d(x), d(y)\} \geq 2k$ for any pair of nonadjacent vertices $\{x, y\} \subseteq V(G)$, then G contains k disjoint cycles.*

The sharpness of the "$\sigma_2(G)$" condition of Theorem 1.1 implies that Fan-type condition of Theorem 1.3 is also sharp.

Other results on disjoint cycles with specified lengths can be found in [1,2,4,9,10,12,13].

This paper is organized as follows: In Section 2, useful lemmas are given and in Section 3, Theorem 1.3 is proved.

2. Lemmas

In the following, let G be a graph with at least $4k$ vertices. If C is a cycle of G, then let $l(C)$ denote the length of C. We shall use the following lemmas to prove our main theorem.

Lemma 2.1 ([11]). *Let $C \subseteq G$ be a cycle of length at least 4 and x a vertex of G not on C. If $d_C(x) \geq 2$, then $G[V(C) \cup \{x\}]$ contains a cycle of length less than $l(C)$ unless $d_C(x) = 2$, $l(C) = 4$ and x is adjacent to two nonadjacent vertices of C.*

Lemma 2.2 ([11]). *Let T be a triangle of G. Let x and y be two distinct vertices of G not on T. Suppose that $d_T(x) + d_T(y) \geq 5$. Then T has a vertex z such that $T - z + x$ is a triangle and $yz \in E(G)$.*

Let P be a path or a cycle of G. If x_i and x_j are two vertices on P, then x_iPx_j denotes the subpath of P with two endvertices x_i and x_j. The following lemma is our main lemma.

Lemma 2.3. *Let s and t be two integers with $s \geq 4$ and $t \leq s + 1$. Suppose that $P \subseteq G$ is a path of order s and $C \subseteq G$ is a cycle of order t such that P and C are disjoint. If $\sum_{x \in V(P)} d_C(x) \geq 2s + 1$, then $G[V(P \cup C)]$ contains two disjoint cycles.*

Proof. Let $P = x_1x_2 \cdots x_s$ and $C = y_1y_2 \cdots y_ty_1$. First suppose that $t \leq s$. Then $\sum_{x \in V(P)} d_C(x) \geq 2s+1 \geq 2t+1$. It is easy to see that there are two vertices $x_i \in V(P)$ and $y_j \in V(C)$ such that $d_C(x_i) \geq 3$ and $d_P(y_j) \geq 3$. Suppose that $\{x_{i_1}, x_{i_2}, x_{i_3}\} \subseteq N_P(y_j)$, where $i_1 < i_2 < i_3$. If $G[V(P \cup C)]$ does not contain two disjoint cycles, then $x_i = x_{i_2}$ and no other vertex of P has three neighbours on C and $d_C(x_i) = d_P(y_j) = 3$. Let $\{y_{j_1}, y_j, y_{j_2}\} = N_C(x_{i_2})$. As $\sum_{x \in V(P)} d_C(x) \geq 2s + 1$, we see that $d_C(x) = 2$ for all $x \in V(P) - \{x_{i_2}\}$. Since $s \geq 4$, we may choose a vertex $z \in V(P) - \{x_{i_1}, x_{i_2}, x_{i_3}\}$ and suppose $N_C(z) = \{u, v\}$. If $y_j \in \{u, v\}$, let $P_1 = y_{j_1}Cy_{j_2}$ such that $y_j \notin V(P_1)$ and $P_1' = zPw_1$, where $w_1 \in \{x_{i_1}, x_{i_3}\}$ such that $x_{i_2} \notin V(P_1')$, then $x_{i_2}P_1x_{i_2}$ and $y_jP_1'y_j$ are two disjoint cycles as in Fig. 1, a contradiction. If $y_j \notin \{u, v\}$, let $P_2 = uCv$ such that $y_j \notin V(P_2)$ and $P_2' = x_{i_2}Pw_2$, where $w_2 \in \{x_{i_1}, x_{i_3}\}$ such that $z \notin V(P_2')$, then zP_2z and $y_jP_2'y_j$ are two disjoint cycles as in Fig. 2, a contradiction.

If $t = s + 1$, then $\sum_{x \in V(P)} d_C(x) \geq 2s + 1 = 2t - 1$. It is easy to see that there is a vertex $x_i \in V(P)$ such that $d_C(x_i) \geq 3$. Suppose that $\{y_{j_1}, y_{j_2}, y_{j_3}\} \subseteq N_C(x_i)$ with $j_1 < j_2 < j_3$. If $G[V(P \cup C)]$ does not contain two disjoint cycles, then there are at most two vertices in P such that each of them has at least 3 neighbours in C. Furthermore, if $d_C(x_i) \geq 3$ and $d_C(x_j) \geq 3$, then $d_C(x_i) = d_C(x_j) = 3$ and $N_C(x_i) = N_C(x_j) = \{y_{j_1}, y_{j_2}, y_{j_3}\}$. In this case, since $\sum_{x \in V(P)} d_C(x) \geq 2s + 1$, we obtain

$$\sum_{x \in V(P)-\{x_i, x_j\}} d_C(x) \geq 2s + 1 - 3 - 3 = (s - 2) + (s - 3).$$

Thus there is a vertex $u \in V(P)-\{x_i, x_j\}$ such that $d_C(u) = 2$ as $s \geq 4$. Assume $N_C(u) = \{v, w\}$. By symmetry, assume u is on the subpath x_iPx_s of P. If $\{v, w\} \nsubseteq \{y_{j_1}, y_{j_2}, y_{j_3}\}$, say $w \notin \{y_{j_1}, y_{j_2}, y_{j_3}\}$. Let $P_1 = y_{j_1}Cy_{j_2}$ and $P_2 = y_{j_3}Cw$ such that $V(P_1) \cap V(P_2) = \emptyset$. Then $G[V(P \cup C)]$ contains two disjoint cycles $x_iP_1x_i$ and uPx_jP_2u (Fig. 3), a contradiction. So $\{v, w\} \subseteq \{y_{j_1}, y_{j_2}, y_{j_3}\}$, say $uy_{j_2} \in E(G)$. Let $P_1 = y_{j_1}Cy_{j_2}$ such that $y_{j_3} \notin V(P_1)$. Then $G[V(P \cup C)]$ contains two disjoint cycles $x_iP_1x_i$ and $y_{j_2}uPx_jy_{j_2}$ (Fig. 3′), again a contradiction. So $d_C(x) \leq 2$ for all $x \in V(P) - \{x_i\}$.

If $5 \leq d_C(x_i) \leq t = s + 1$, then

$$\sum_{x \in V(P)-\{x_i\}} d_C(x) \geq 2s + 1 - (s + 1) = (s - 1) + 1,$$

which implies that there is a vertex $x_j \in V(P) - \{x_i\}$ such that $d_C(x_j) = 2$. Assume $N_C(x_i) \supseteq \{y_{j_1}, y_{j_2}, \ldots, y_{j_5}\}$ and $N_C(x_j) = \{u, v\}$. Then $G[V(P \cup C)]$ contains two disjoint cycles as in Fig. 4, a contradiction. If $3 \leq d_C(x_i) \leq 4$, then

$$\sum_{x \in V(P)-\{x_i\}} d_C(x) \geq 2s + 1 - 4 = (s - 1) + (s - 2).$$

Thus there are two vertices $x_{i_1}, x_{i_2} \in V(P) - \{x_i\}$ such that $d_C(x_{i_1}) = d_C(x_{i_2}) = 2$ since $s \geq 4$. Moreover, $N_C(x_{i_1}) = N_C(x_{i_2}) \subseteq \{y_{j_1}, y_{j_2}, y_{j_3}\}$. Again

$$\sum_{x \in V(P)-\{x_{i_1}, x_i, x_{i_2}\}} d_C(x) \geq 2s + 1 - 4 - 4 = (s - 3) + (s - 4),$$