Discrete Mathematics 341 (2018) 772–780

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

On the excess of vertex-transitive graphs of given degree and girth

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ARTICLE INFO

Article history:
Received 25 November 2016
Received in revised form 17 November 2017
Accepted 20 November 2017

Keywords:
Cage problem
Vertex-transitive graph
Lower bound
Degree
Girth

ABSTRACT

We consider a restriction of the well-known Cage Problem to the class of vertex-transitive graphs, and consider the problem of finding the smallest vertex-transitive k-regular graphs of girth g. Counting cycles to obtain necessary arithmetic conditions on the parameters (k, g), we extend previous results of Biggs, and prove that, for any given excess e and any given degree $k \geq 4$, the asymptotic density of the set of girths g for which there exists a vertex-transitive (k, g)-cage with excess not exceeding e is 0.

1. Introduction

The main focus of our paper is on the orders of k-regular graphs of girth g. To simplify our notation, we shall refer to these graphs as the *(k,g)-graphs*. The *Cage Problem* is the problem which calls for finding a *smallest k-regular graph* of *girth* g, called a *(k,g)-cage*. If we denote the order of a (k, g)-cage by $n(k, g)$, the order of any (k, g)-graph is greater or equal to $n(k, g)$. The Cage Problem was first considered by Tutte [13]. A well known lower bound on the order $|V(G)|$ of a k-regular graph G of girth g, called the *Moore bound* and denoted by $M(k, g)$, depends on the parity of g:

$$|V(G)| \geq n(k,g) \geq M(k,g) = \begin{cases} 1 + \displaystyle\sum_{i=0}^{(g-3)/2} k(k-1)^i = \frac{k(k-1)^{(g-1)/2} - 2}{k-2}, & g \text{ odd}, \\ 2 \displaystyle\sum_{i=0}^{(g-2)/2} (k-1)^i = \frac{2(k-1)^{g/2} - 2}{k-2}, & g \text{ even}. \end{cases}$$

Graphs for which $|V(G)| = M(k, g)$ are necessarily cages and are called *Moore graphs*. The existence of a (k, g)-cage for any pair (k, g) was first shown by Erdös and Sachs [12,4], but the exact orders $n(k, g)$ are only known for very limited sets of parameter pairs [5]. The *excess* $e(G)$ of a k-regular graph G of girth g is the difference between its order and the corresponding value of the Moore bound, hence, $|V(G)| = M(k, g) + e(G)$, for any (k, g)-graph G.

Many of the cages as well as the smallest known (k, g)-graphs turn out to be *vertex-transitive* [5], i.e., graphs whose automorphism groups act transitively upon their vertices. The reason for such frequent occurrence among the smallest (k, g)-graphs is not well understood, but one of the reasons might lie in the fact that vertex-transitive graphs are locally

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https://doi.org/10.1016/j.disc.2017.11.015

isomorphic around each vertex, and hence each of their vertices lies on cycles of the same lengths. This seems to be a feature shared by the extreme (k, g)-graphs as well. Based on this observation, considering a restricted version of the original cage problem and looking for smallest vertex-transitive (k, g)-graphs (which we shall refer to as *vertex-transitive cages*) and their corresponding orders $vt(k, g)$ will most likely lead to improvements in our understanding of both the general Cage Problem and the structure of vertex-transitive graphs. Obviously, $vt(k, g) \geq n(k, g)$.

The existence of vertex-transitive (k, g)-graphs for any pair $k, g \geq 3$ has been established for example in [9]. In the case of general cages, the question of whether there exists a universal bound on the excess is still open. On the other hand, in the more specialized case of vertex-transitive cages, this question has been answered in negative, and the excess of vertex-transitive (k, g)-cages can be arbitrarily large. This result is due to Biggs:

Theorem 1.1 (*[2]*)**.** *For each odd integer $k \geq 3$, there is an infinite sequence of values of g such that the excess e of any vertex-transitive graph of degree k and girth g satisfies $e > \frac{g}{k}$.*

In our paper, we show that Biggs' result [2] holds not only for infinitely many g's, but, in fact, holds for almost all g's for any given $k \geq 4$. More specifically, we show that for any given excess e and degree $k \geq 4$, the set of g's for which $vt(k, g) - M(k, g) < e$ is of asymptotic density 0 (when compared to the set of all girths $g \geq 3$).

The main technique used in this paper depends on counting cycles in graphs whose orders are close to the Moore bound. In 1971 Friedman [7] used this technique to show that Moore graphs for certain parameter pairs (k, g) cannot exist. It is known that the number of cycles of length l passing through a given vertex in a vertex-transitive graph is independent of the choice of the vertex (while no such result has been proved for cages in general). Let G be a graph, let b be a vertex of G, and let $n \geq 3$ be an integer. By $\mathbf{c}_G(b, n)$ we denote *the number of n-cycles in G that contain b*. Our counting techniques rely on the following fairly obvious lemma.

Lemma 1.2 (*[6]*)**.** *If G is a vertex-transitive graph and $n \geq 3$ is a positive integer, then the following hold:*

(1) $\mathbf{c}_G(x, n) = \mathbf{c}_G(y, n)$, *for all $x, y \in V(G)$;*
(2) *n divides the product $\mathbf{c}_G(x, n) \cdot |V(G)|$, for all $x \in V(G)$.*

In addition to obtaining the above stated density results, in the last two sections of our paper we address the question of the magnitude of the excess. As the odd- and even-girth cases differ in several important characteristics, we derive our results separately for odd- and even-girth regular graphs.

2. The excess of vertex-transitive graphs of odd girth

In this section we prove that for any fixed pair $k \geq 4$, $e \geq 1$, the excess for (k, g)-vertex-transitive cages exceeds e for almost all odd girths g. Our arguments are analogous to those used in [6] in which the authors show similar density results in the so-called Degree/Diameter Problem [10]. This problem, often considered dual to the Cage Problem, calls for determining the largest graphs of given degree and diameter.

We begin by presenting an upper and lower bound on the number of g-cycles containing a fixed vertex v in a general (i.e., not necessarily vertex-transitive) (k, g)-graph.

Lemma 2.1. *Let G be a (k, g)-graph of degree $k \geq 3$, odd girth g, excess $e(G) \geq 1$, and let v be an arbitrary vertex of G. The number $\mathbf{c}_G(v, g)$ of g-cycles containing v satisfies the following lower and upper bounds:*

$$\frac{k(k-1)^{(g-1)/2}}{2} - \frac{ek}{2} \leq \mathbf{c}_G(v, g) \leq \frac{k(k-1)^{(g-1)/2}}{2}. \quad (1)$$

Proof. If G were a Moore graph, the number of g-cycles through a fixed vertex v would satisfy the identity $\mathbf{c}_G(v, g) = \frac{1}{2}k(k-1)^{(g-1)/2}$, proved in [7]. In order to prove the bounds for graphs G whose orders exceed the Moore bound $M(k, g)$, i.e., graphs with excess $e(G) \geq 1$, we introduce the following notation from [1]. Let v be an arbitrary vertex of G, and let $N_G^i(v)$ denote the set of vertices in G whose distance from v is equal to i. Let $g = 2t + 1$, and let $\mathcal{T}_v^G$ be the subgraph of G whose vertices belong to the union $\bigcup_{i=0}^t N_G^i(v)$ and whose edges are the edges of the subgraph of G induced by the subset $\bigcup_{i=0}^t N_G^i(v)$ *minus* the edges with both ends belonging to the last layer $N_G^t(v)$. Since G contains no cycles shorter than g, it is easy to see that $\mathcal{T}_v^G$ is a tree of order $M(k, g)$; we will refer to this tree as the *Moore tree of G with respect to v*. In addition, we will call the edges connecting the vertices from $N_G^t(v)$ (and thus excluded from $\mathcal{T}_v^G$) *edges horizontal with respect to v*. The key observation of our argument relates the number of g-cycles through v with the number of edges horizontal with respect to v. More specifically, since $\mathcal{T}_v^G$ is a tree of depth t rooted at v, any $g = (2t+1)$-cycle through v must consist of two edge-disjoint paths of length t connecting v to vertices u, w in $N_G^t(v)$ and a single horizontal edge connecting u and w. Consequently, $\mathbf{c}_G(v, g)$ is equal to the number of edges horizontal with respect to v. This yields the upper bound $\mathbf{c}_G(v, g) \leq \frac{k(k-1)^{(g-1)/2}}{2}$, where the right side of the inequality is the maximal possible number of horizontal edges—the number of vertices in $N_G^t(v)$ multiplied by $(k-1)$ and divided by 2 (each edge is counted twice this way). Let X_v now be the *excess set of G* with respect to v, i.e., the set of e vertices of G that do not belong to $\mathcal{T}_v^G$. Note that the only vertices from $\mathcal{T}_v^G$ the vertices from X_v might be connected to are the vertices in $N_G^t(v)$. Since the e vertices in X_v are of degree k, the maximum number of edges between