Contents lists available at ScienceDirect
Discrete Mathematics
journal homepage: www.elsevier.com/locate/disc

Domino tilings of the expanded Aztec diamond☆

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ARTICLE INFO

Article history:
Received 28 February 2016
Received in revised form 14 October 2017
Accepted 17 October 2017
Available online xxxx

Keywords:
Aztec diamond
Domino tiling
Dimer covering
Perfect matching

ABSTRACT

The expanded Aztec diamond is a generalized version of the Aztec diamond, with an arbitrary number of long columns and long rows in the middle. In this paper, we count the number of domino tilings of the expanded Aztec diamond. The exact number of domino tilings is given by recurrence relations of state matrices by virtue of the state matrix recursion algorithm, recently developed by the author to solve various two-dimensional regular lattice model enumeration problems.

1. Introduction

In both combinatorial mathematics and statistical mechanics, domino tiling of the Aztec diamond is an important subject. The Aztec diamond of order n consists of all lattice squares that lie completely inside the diamond shaped region $\{(x, y) : |x| + |y| \leq n + 1\}$. The Aztec diamond theorem from the excellent article of Elkies, Kuperberg, Larsen and Propp [1] states that the number of domino tilings of the Aztec diamond of order n is equal to $2^{n(n+1)/2}$. From the statistical mechanics viewpoint, tilings of large Aztec diamonds exhibit a striking feature. The Arctic circle theorem proved by Jockusch, Propp and Shor [4] says that a random domino tiling of a large Aztec diamond tends to be frozen outside a certain circle.

The augmented Aztec diamond looks much like the Aztec diamond, except that there are three long columns in the middle instead of two. See Fig. 1. The number of domino tilings of the augmented Aztec diamond of order n was computed by Sachs and Zernitz [14] as $\sum_{k=0}^{n} \binom{n}{k} \cdot \binom{n+k}{k}$, known as the Delannoy numbers. Notice that the former number is much larger than the later. Indeed, the number of domino tilings of a region is very sensitive to boundary conditions [6,7]. More interesting patterns related to the Aztec diamond allowing some squares removed have been deeply studied and Propp proposed a survey of these works [13].

In this paper, we consider a generalized region of the Aztec diamond which has an arbitrary number of long columns and long rows in the middle. The expanded (p, q)-*Aztec diamond* of order n, denoted by $AD_{(p,q;n)}$, is defined as the union of $2n(n + p + q + 1) + pq$ unit squares, arranged in bilaterally symmetric fashion as a stack of $2n + q$ rows of squares, the rows having lengths $p+2, p+4, \dots, 2n+p-2, 2n+p, \dots, 2n+p, 2n+p-2, \dots, p+2$, as drawn in Fig. 2. Let $\alpha_{(p,q;n)}$ denote the number of domino tilings of $AD_{(p,q;n)}$. Note that $\alpha_{(p,q;n)} = 0$ for odd pq because $AD_{(p,q;n)}$ consists of odd number of squares.

Recently several important enumeration problems regarding various two-dimensional regular lattice models are solved by means of the *state matrix recursion algorithm*, introduced by the author. This algorithm provides recursive matrix-relations to enumerate monomer and dimer coverings and independent vertex sets known as the Merrifield–Simmons index. These problems have been major outstanding unsolved combinatorial problems, and this algorithm shows considerable promise for further two-dimensional lattice model enumeration studies. See [8,9,12] for more details.

☆ This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIP) (No. NRF-2017R1A2B2007216).
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https://doi.org/10.1016/j.disc.2017.10.016

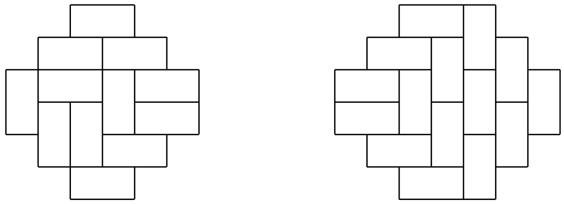

Fig. 1. The Aztec diamond of order 3, the augmented Aztec diamond of order 3, and their domino tilings.

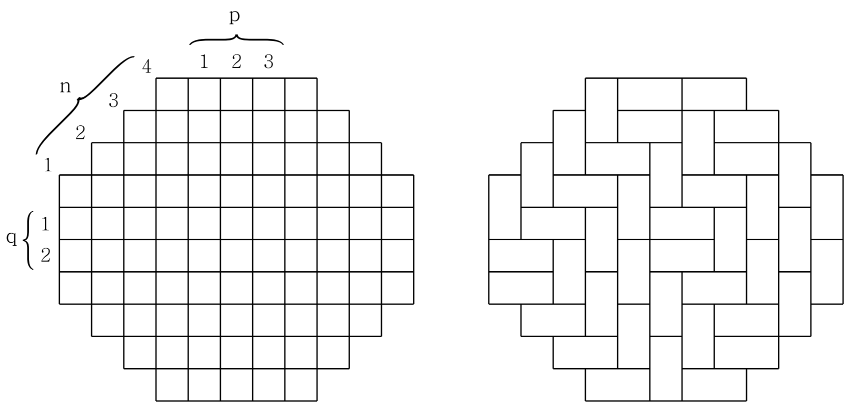

Fig. 2. The expanded Aztec diamond $AD_{(3,2;4)}$ and a domino tiling.

Using the state matrix recursion algorithm, we present a recursive formula producing the exact number of $\alpha_{(p,q;n)}$. Throughout the paper, $\mathbb{O}$ is a zero-matrix with an appropriate size, and A^t is the transpose of a matrix A.

Theorem 1. *The number $\alpha_{(p,q;n)}$ of domino tilings of the (p, q)-Aztec diamond of order n is the $(1, 1)$-entry of the following $2^p \times 2^p$ matrix*

$$\prod_{k=1}^{n} \left[\begin{bmatrix} A_{p+2k-2} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} \quad A_{p+2k-1} \right] \cdot \left(C_{p+2n}\right)^q \cdot \left(\prod_{k=1}^{n} \left[\begin{bmatrix} A_{p+2k-2} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} \quad A_{p+2k-1} \right] \right)^t,$$

where the $2^{k-1} \times 2^k$ matrix A_k and the $2^k \times 2^k$ matrix C_k are defined by

$$A_k = \begin{bmatrix} \begin{bmatrix} A_{k-2} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} & A_{k-1} \\ A_{k-1} & \mathbb{O} \end{bmatrix} \text{ and } C_k = \begin{bmatrix} \begin{bmatrix} C_{k-2} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} & C_{k-1} \\ C_{k-1} & \mathbb{O} \end{bmatrix}$$

with seed matrices $A_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, $C_0 = \begin{bmatrix} 1 \end{bmatrix}$ and $C_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Here, $\begin{bmatrix} 1 & 0 \end{bmatrix}$ is used for the undefined matrix $\begin{bmatrix} A_0 & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix}$ when $p = 0$ and $k = 1$.

We adjust the main scheme of the state matrix recursion algorithm introduced in [9] to solve Theorem 1 in Sections 2–4 as three stages.

2. Stage 1. Conversion to domino mosaics

First stage is dedicated to the installation of the mosaic system for domino tilings on the expanded Aztec diamond region. A mosaic system was invented by Lomonaco and Kauffman to give a precise and workable definition of quantum knots representing an actual physical quantum system [5]. Later, the author et al. have developed a state matrix argument for knot mosaic enumeration in a series of papers [2,3,10,11]. This argument has been developed further into the state matrix recursion algorithm by which we enumerate monomer–dimer coverings on the square lattice [9]. We follow the notion and terminology in the paper with some modifications.

In this paper, we consider the four *mosaic tiles* T_1, T_2, T_3 and T_4 illustrated in Fig. 3. Their side edges are labeled with two letters a and b as follows: letter 'a' if it is not touched by a thick arc on the tile, and letter 'b' for otherwise.

Please cite this article in press as: S. Oh, Domino tilings of the expanded Aztec diamond, Discrete Mathematics (2017), https://doi.org/10.1016/j.disc.2017.10.016.