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Minimizing the number of independent sets in triangle-free regular graphs

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ABSTRACT

Recently, Davies, Jenssen, Perkins, and Roberts gave a very nice proof of the result (due, in various parts, to Kahn, Galvin–Tetali, and Zhao) that the independence polynomial of a d-regular graph is maximized by disjoint copies of $K_{d,d}$. Their proof uses linear programming bounds on the distribution of a cleverly chosen random variable. In this paper, we use this method to give lower bounds on the independence polynomial of regular graphs. We also give a new bound on the number of independent sets in triangle-free cubic graphs.

1. Introduction

Extremal problems involving the number of substructures of a graph of a given type have popped up in quite a few different contexts of late. One of the best known such results is due to Kahn [6] and Zhao [11]. We let $\mathrm{Ind}(G)$ be the set of independent sets in a graph G. Their theorem bounds $\mathrm{ind}(G) = |\mathrm{Ind}(G)|$ for regular graphs.

Theorem 1 (*Kahn, Zhao*). *If G is a d-regular graph on n vertices, then*

$$\mathrm{ind}(G)^{1/n} \leq \mathrm{ind}(K_{d,d})^{1/2d}.$$

One source for questions of this type is the field of statistical mechanics. For instance, the *hard-core model* on a graph G is a probability distribution on the independent sets of G in which a independent set I is chosen with probability proportional to $\lambda^{|I|}$. Here $\lambda > 0$ is a parameter called the *fugacity*. The normalizing factor is

$$P_G(\lambda) = \sum_{I \in \mathrm{Ind}(G)} \lambda^{|I|},$$

known to graph theorists as the *independence polynomial of G* and to statistical physicists as the *partition function* of this hard-core model.

Kahn [6] in fact proved the analogue of Theorem 1 for the independence polynomial of bipartite graphs with fugacity $\lambda \geq 1$, i.e.,

$$P_G(\lambda)^{1/n} \leq P_{K_{d,d}}(\lambda)^{1/2d}.$$

Galvin and Tetali [5] extended Kahn's result to cover the case $0 < \lambda < 1$. Finally, Zhao [11] proved the full theorem using a clever lifting argument.

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More recently, Davies, Jenssen, Perkins, and Roberts [3] gave an independent proof introducing an audacious new approach utilizing linear programming. Following Davies et al., we will derive bounds on $P_G(\lambda)$ by considering the *occupancy fraction*, denoted $\alpha_G(\lambda)$. This is the expected fraction of vertices of G belonging to a random independent set chosen according the hard-core model. More explicitly,

$$\alpha_G(\lambda) = \frac{1}{n}\frac{\sum_{I\in \mathrm{Ind}(G)}|I|\lambda^{|I|}}{P_G(\lambda)} = \frac{1}{n}\frac{\lambda P_G'(\lambda)}{P_G(\lambda)}.$$

Davies et al. [3] proved the following.

Theorem 2 (*Davies, Jenssen, Perkins, Roberts*)**.** *For all d-regular graphs G and all $\lambda > 0$, it is the case that*

$$\alpha_G(\lambda) \le \alpha_{K_{d,d}}(\lambda).$$

Because $\alpha_G(\lambda)$ is essentially the logarithmic derivative of $P_G(\lambda)$, this is a strengthening of Theorem 1. The proof of Theorem 3 below shows how to use the occupancy fraction to bound the independence polynomial.

In this paper, we investigate lower bound analogues of Theorem 2. In Section 2, we give an example of the linear programming method by proving that for any d-regular graph, the occupancy fraction is bounded below by that of K_{d+1}. As pointed out by Davies et al. [4], this result can also be deduced from the proof for the lower bound on $\mathrm{ind}(G)$ in graphs with maximum degree at most d proved by the authors in [2].

In the final section, we discuss a problem raised by Kahn [7], that of giving a lower bound on $P_G(\lambda)$ for d-regular triangle-free graphs. We use the same occupancy fraction approach to give bounds in this case. In Zhao's lovely survey article on this area [12, Problem 9.5], he proposes to study the general problem of finding the infimum and supremum of $\mathrm{ind}(G)^{1/n(G)}$ over d-regular graphs not satisfying a given excluded subgraph condition.

When λ is large, the hard-core model is biased strongly towards large independent sets. Indeed,

$$\lim_{\lambda\to\infty} \alpha_G(\lambda) = \frac{\alpha(G)}{n}, \tag{1}$$

where $\alpha(G)$ is the *independence number of* G, and the ratio $\alpha(G)/n$ is the *independence ratio of* G. In Section 3, we focus on triangle-free cubic graphs. Here we are able to give a bound that is relatively good when $\lambda = 1$. We conjectured that the Petersen graph is extremal when $\lambda = 1$, which was recently proved by Perarnau and Perkins [8]. Indeed, they prove that the occupancy fraction for cubic triangle-free graphs is minimized by the Petersen graph for $0 < \lambda \le 1$. This cannot be extended to all λ since the independence ratio for triangle-free cubic graphs is minimized by GP(7, 2), a generalized Petersen graph (see Staton [10]). In fact, we conjecture that for any λ, $P_G(\lambda)$ is minimized by either the Petersen graph or the generalized Petersen graph.

2. Lower bounds for the hard-core model on regular graphs

In this section, we present a proof of a best possible lower bound on the occupancy fraction for d-regular graphs. The proof we give serves as an introduction to the linear programming method of Davies et al. [3]. In a subsequent paper, Davies et al. [4] observe that this result follows relatively straightforwardly from a result of the current authors in [2]. Our proof appears at the end of this section after a number of lemmas concerning the linear programming approach.

Theorem 3. *If G is a d-regular graph on n vertices and $\lambda > 0$, then*

$$\alpha_G(\lambda) \ge \alpha_{K_{d+1}}(\lambda).$$

As a consequence, we have

$$P_G(\lambda)^{1/n} \ge P_{K_{d+1}}(\lambda)^{1/(d+1)}.$$

Equality is, in both cases, only achieved for G a disjoint union K_{d+1}s.

Following Davies et al. [3], we will consider, for each vertex in V, the probability that it belongs to, and the probability that it is covered by, a randomly chosen independent set. To be explicit, if I is an independent set, we say that $v \in V(G)$ is *occupied* if $v \in I$ and *uncovered* if $I \cap N(v) = \emptyset$. If I is distributed according to the hard-core model, we write p_v for $\mathbb{P}(v \in I)$ and q_v for $\mathbb{P}(v \text{ is uncovered})$. Note that both p_v and q_v are functions of λ. Also, it is the case that $p_v \le q_v$ since $\{I \in \mathrm{Ind}(G) : v \text{ is occupied}\} \subseteq \{I \in \mathrm{Ind}(G) : v \text{ is uncovered}\}$.

Lemma 4. *In the hard-core model on G with fugacity $\lambda > 0$, we have*

(1) $p_v = \frac{\lambda}{1+\lambda} q_v$, *and*
(2) $\alpha_G(\lambda) = \frac{1}{n}\sum_{v\in V(G)} p_v$.