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Global forcing number for maximal matchings

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ABSTRACT

Let $\mathcal{M}(G)$ denote the set of all maximal matchings in a simple graph G, and $f : \mathcal{M}(G) \to \{0, 1\}^{|E(G)|}$ be the characteristic function of maximal matchings of G. Any set $S \subseteq E(G)$ such that $f|_S$ is an injection is called a global forcing set for maximal matchings in G, and the cardinality of smallest such S is called the global forcing number for maximal matchings of G. In this paper we establish sharp lower and upper bounds on this quantity and prove explicit formulas for certain classes of graphs. At the end, we also state some open problems and discuss some further developments.

1. Introduction and motivation

The concept of **forcing set** is one of many graph-theoretical concepts whose origins can be traced back to the study of resonance structures in mathematical chemistry where it was introduced under the name of the innate degree of freedom [10,12]. Later it attracted significant attention also in purely graph-theoretical literature [1,2,14,15,23]. The forcing sets were first defined locally, with reference to particular Kekulé structures (or perfect matchings in mathematical literature), and global results were obtained by considering extremal values over the set of all relevant structures. Then the focus shifted to the study of forcing sets that were defined globally in a graph, motivated by the need to efficiently code and manipulate perfect matchings in large-scale computations [19,20]. It turned out that many results could be successfully transferred from the local to the global context. In particular, explicit formulas for the global forcing number for some benzenoid graphs, rectangular and triangular grids and complete graphs were obtained by some of the present authors [5,16,18,21].

Instrumental in obtaining those results were the elements of well-developed structural theory available for perfect matchings. No such theory, however, exists for much less researched but still very useful and interesting class of large matchings, known as maximal matchings. Hence, we were unable to simply transfer the above results when a need for analogous concepts arose in course of our work on maximal matchings. The aim of this paper is to fill the gap by extending the concepts of global forcing set and global forcing number also to maximal matchings and to obtain results analogous to those mentioned for the perfect matching case.

The paper is organized as follows. In the next section we define the terms relevant for our subject and present some preliminary results. Section 3 contains some lower bounds on the global forcing number and also a monotonicity results used later. Sections 4 and 5 present results on trees and complete graphs, respectively, while in Section 6 we present bounds

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for graphs of a given cyclomatic number. Finally, in the concluding section we comment on some open problems and indicate some possible directions for future research.

2. Definitions and preliminary results

All graphs in this paper are tacitly supposed to be simple and connected unless explicitly stated otherwise. Let G be a graph with set of vertices $V(G)$ and set of edges $E(G)$. We will denote by $n = |V(G)|$ the number of vertices and by $m = |E(G)|$ the number of edges in G. As usual, the path, the star, and the complete graph on n vertices are denoted by P_n, S_n and K_n, respectively.

Let G be a graph and H be any subgraph of G. We denote by $G - H$ the graph obtained by deleting from G all vertices of H and all edges incident with them. If S is a set of edges of G, then $G - S$ denotes the graph obtained from G by removing edges from S without removing their end-vertices. We reserve notation $S \setminus M$ for the set difference of two sets of edges.

A connected graph G is **acyclic** or a **tree** if G does not contain cycles. If G contains exactly one cycle, we say G is an **unicyclic** graph. Finally, for any graph G we define its **cyclomatic number** $c(G)$ by $c(G) = |E(G)| - |V(G)| + 1$. That is the smallest number of edges one must remove from a graph to obtain a tree. If G is a tree, then $c(G) = 0$, and if G is an unicyclic graph, then $c(G) = 1$. A vertex u in a graph G is a **leaf** if u has exactly one neighbor. The only neighbor of a leaf in G is called a **petal**.

A **matching** in a graph G is any set of edges $M \subseteq E(G)$ such that every vertex in G is incident with at most one edge from M. The number of edges in M is called its **size**. Matchings of small size are quite uninteresting, since they are easy to construct and enumerate. On the other hand, "large" matchings serve as models for many problems in which we have entities capable of interactions over a given connection pattern. Whenever one neighbor can monopolize all interaction capability of an entity, rendering it unavailable for its other neighbors, matchings naturally appear, and existence of "large" matchings is usually desirable as it signals good efficiency of the underlying process. Hence, we are interested in study of large matchings.

A matching M is **maximum** if there is no matching in G of a greater size. The cardinality of any maximum matching in G is called the **matching number** of G and denoted by $\nu(G)$. Since each edge of a matching saturates two vertices of G, no matching in G can have size greater than $\lfloor n/2 \rfloor$. We say that a matching M is **perfect** if every vertex from G is incident with exactly one edge from M. Obviously, only graphs on an even number of vertices can have perfect matchings. If a graph G on an odd number of vertices has $\nu(G) = \lfloor n/2 \rfloor$, we say that G has an **almost perfect matching**.

Another way of measuring how large is a given matching is based on (im)possibility of its extension to a larger matching. We say that matching M is **maximal** if there is no matching M' in G such that $M \subset M'$. Note that every maximum matching in G is also maximal, but the opposite is, in general, not true. Maximal matchings usually come in different sizes. The smallest size of a maximal matching in G is called the **saturation number** of G and denoted by $s(G)$. The largest size is, of course, the matching number $\nu(G)$. If all maximal matchings in G are of the same size (and hence maximum), graph G is **equimatchable**.

There is a marked asymmetry in the way maximum and maximal matchings are studied and represented in the literature. While the maximum (and in particular perfect) matchings are well researched and understood (see, for example, monographs [13] and [4]), results on their maximal counterparts are much less abundant. We mention here some papers dealing with maximal matchings in trees [11,22], with equimatchable graphs [8,9], and two recent papers coauthored by one of the present authors about structural and enumerative aspects of maximal matchings in linear polymers [6,7]. One of possible reasons for the scarcity of results might be that, at the moment, there is no structural theory for maximal matchings analogous to the one available for maximum matchings.

Any non-maximal matching can be extended to a maximal matching. In particular, for any edge $e \in E(G)$ there is a maximal matching M containing e. This stands in sharp contrast with the situation for perfect matchings, where such property (1-extendability) imposes strong structural conditions on G. The idea of finding a subset of a perfect matching which is in a unique way extendable to the whole matching gave rise to the concept of forcing set.

For a given perfect matching M in G, *its* forcing set is defined as any subset of M that is not contained in any other perfect matching of G. The forcing number of a perfect matching M was defined as the size of any smallest forcing set of M. Note that forcing sets and numbers are defined for each perfect matching of G. The idea was generalized to global setting in two different ways. One was to study extremal forcing sets and forcing numbers over all perfect matchings; the other was to look for subsets of edges of G, not necessarily matchings, such that no two perfect matchings coincide on them. The later approach gave rise to the concept of global forcing sets and numbers for perfect matchings. Now we extend the idea also to maximal matchings.

A **global forcing set for maximal matchings** of a graph G is any set $S \subseteq E(G)$ such that $M_1\big|_S \neq M_2\big|_S$ for any two maximal matchings M_1 and M_2. Any global forcing set for maximal matchings in G of the smallest cardinality is called a minimum global forcing set and its cardinality, denoted by $\varphi_{gm}(G)$, is called the **global forcing number for maximal matchings** in G. Throughout the rest of the paper we will say only global forcing set (or number) of graph G tacitly assuming it is a global forcing set (or number) for maximal matchings in G unless explicitly stated otherwise.

Global forcing sets in a given graph G have an obvious monotonicity property.

Proposition 1. *If $S \subset E(G)$ is a global forcing set, then each $S' \supset S$ is also a global forcing set. If $S \subset E(G)$ is not a global forcing set, then no $S' \subset S$ can be a global forcing set.*