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Heights of minor 5-stars in 3-polytopes with minimum degree 5 and no vertices of degree 6 and 7[☆]

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ABSTRACT

Given a 3-polytope P, by $h(P)$ we denote the minimum of the maximum degrees (height) of the neighborhoods of 5-vertices (minor 5-stars) in P.

In 1940, H. Lebesgue gave an approximate description of the neighborhoods of 5-vertices in the class $\mathbf{P}_5$ of 3-polytopes with minimum degree 5.

In 1996, S. Jendrol' and T. Madaras showed that if a polytope P in $\mathbf{P}_5$ is allowed to have a 5-vertex adjacent to four 5-vertices (called a minor $(5, 5, 5, 5, \infty)$-star), then $h(P)$ can be arbitrarily large.

For each P^* in $\mathbf{P}_5$ with neither vertices of degree 6 or 7 nor minor $(5, 5, 5, 5, \infty)$-star, it follows from Lebesgue's theorem that $h(P^*) \leq 23$.

We prove that every such polytope P^* satisfies $h(P^*) \leq 14$, which bound is sharp. Moreover, if 6-vertices are allowed but 7-vertices forbidden, or vice versa, then the height of minor 5-stars in $\mathbf{P}_5$ under the absence of minor $(5, 5, 5, 5, \infty)$-stars can reach 15 or 17, respectively.

1. Introduction

The degree of a vertex or face x in a convex finite 3-dimensional polytope (called a *3-polytope*) is denoted by $d(x)$. A *k-vertex* is a vertex v with $d(v) = k$. A *k^+-vertex* (*k^--vertex*) is one of degree at least k (at most k). Similar notation is used for the faces. A 3-polytope with minimum degree δ is denoted by P_δ. The *weight* of a subgraph S of a 3-polytope is the sum of degrees of the vertices of S in the 3-polytope. The *height* of a subgraph S of a 3-polytope is the maximum degree of the vertices of S in the 3-polytope. A k-star, a star with k rays, $S_k(v)$ is *minor* if its center v has degree at most 5. In particular, the neighborhoods of 5-vertices are minor 5-stars and vice versa. All stars considered in this note are minor. By $w(S_k)$ and $h(S_k)$ we denote the minimum weight and height, respectively, of minor k-stars in a given 3-polytope.

In 1904, Wernicke [19] proved (in dual form) that every P_5 has a 5-vertex adjacent to a 6^--vertex. This result was strengthened by Franklin [13] in 1922 to the existence of a 5-vertex with two 6^--neighbors. In 1940, in attempts to solve the Four Color Problem, Lebesgue [17, p. 36] gave an approximate description of the neighborhoods of 5-vertices in P_5's. In particular, this description implies the results in [19,13] and shows that there is a 5-vertex with three 7^--neighbors.

For P_5's, the bounds $w(S_1) \leq 11$ (Wernicke [19]) and $w(S_2) \leq 17$ (Franklin [13]) are tight. It was proved by Lebesgue [17] that $w(S_3) \leq 24$, which was improved in 1996 by Jendrol' and Madaras [15] to the sharp bound $w(S_3) \leq 23$. Furthermore, Jendrol' and Madaras [15] gave a precise description of minor 3-stars in P_5's. Lebesgue [17] proved $w(S_4) \leq 31$, which was strengthened by Borodin and Woodall [12] to the tight bound $w(S_4) \leq 30$. Note that $w(S_3) \leq 23$ easily implies $w(S_2) \leq 17$

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and immediately follows from $w(S_4) \leq 30$ (in both cases, it suffices to delete a vertex of maximum degree from a minor star of the minimum weight). Recently, Borodin and Ivanova [5] obtained a precise description of 4-stars in P_5's.

For arbitrary 3-polytopes, that is for P_3's, the following results concerning $(d-2)$-stars at d-vertices, $d \leq 5$, are known. Van den Heuvel and McGuinness [18] proved (in particular) that either $w(S_1(v)) \leq 14$ with $d(v) = 3$, or $w(S_2(v)) \leq 22$ with $d(v) = 4$, or $w(S_3(v)) \leq 29$ with $d(v) = 5$. Balogh et al. [1] proved that there is a 5^--vertex adjacent to at most two 11^+-vertices. Harant and Jendrol' [14] strengthened these results by proving (in particular) that either $w(S_1(v)) \leq 13$ with $d(v) = 3$, or $w(S_2(v)) \leq 19$ with $d(v) = 4$, or $w(S_3(v)) \leq 23$ with $d(v) = 5$. Recently, Borodin and Ivanova [4] obtained a precise description of $(d-2)$-stars in P_3's.

For P_3's, the problem of describing $(d-1)$-stars at d-vertices, $d \leq 5$, called *pre-complete stars*, appears difficult. As follows from the double n-pyramid, the minimum weight $w(S_{d-1})$ of pre-complete stars in P_4's can be arbitrarily large. Even when $w(S_{d-1})$ is restricted by appropriate conditions, the tight upper bounds on it are unknown. Borodin et al. [3,2] proved (in particular) that if a planar graph with $\delta \geq 3$ has no edge joining two 4^--vertices, then there is a star $S_{d-1}(v)$ with $w(S_{d-1}(v)) \leq 38 + d(v)$, where $d(v) \leq 5$ (see [2, Theorem 2.A]). Jendrol' and Madaras [16] proved that if the weight $w(S_1)$ of every edge in a P_3 is at least 9, then there is a pre-complete star of height at most 20, where the bound of 20 is best possible.

The more general problem of describing d-stars at d-vertices, $d \leq 5$, called *complete stars*, at the moment seems untractable for arbitrary 3-polytopes and difficult even for P_5's.

Jendrol' and Madaras [15] showed that if a P_5 has a 5-vertex adjacent to four 5-vertices, called a minor $(5, 5, 5, 5, \infty)$-star, then $h(P_5)$ can be arbitrarily large.

For each P_5 that has neither 6-vertices nor minor $(5, 5, 5, 5, \infty)$-star, it follows from Lebesgue's theorem that $h(P_5) \leq 41$. Recently, this bound was lowered to $h(P_5) \leq 28$ by Borodin, Ivanova, and Jensen [8], then to $h(P_5) \leq 23$ in Borodin and Ivanova [6], and finally to the tight bound $h(P_5) \leq 17$ by Borodin, Ivanova, and Nikiforov [11].

For P_5 having neither minor $(5, 5, 5, 5, \infty)$-stars nor vertices of degree from 6 to 9, Lebesgue's bound $h(P_5) \leq 14$ was recently improved by Borodin and Ivanova [7] to the sharp bound $h(P_5) \leq 12$. The same bound holds under a weaker assumption of the absence of vertices of degree from 6 to 8 (see [10]).

For every P_5 with neither vertices of degree 6 or 7 nor minor $(5, 5, 5, 5, \infty)$-star, it follows from Lebesgue's theorem that $h(P) \leq 23$. The purpose of this note is to prove the following fact.

Theorem 1. *Every 3-polytope P with minimum degree 5 and containing neither 6-vertices nor 7-vertices nor else 5-vertices adjacent to four 5-vertices satisfies $h(P) \leq 14$, which bound is best possible.*

We note that the hypothesis of Theorem 1 cannot be weakened since if 6-vertices are allowed but 7-vertices forbidden, or vice versa, then the height of minor 5-stars in $\mathbf{P_5}$ under the absence of minor $(5, 5, 5, 5, \infty)$-stars can reach 15 or 17, respectively, as shown in [9,11]. Here, 15 and 17 are sharp and improve the corresponding Lebesgue's bounds $h(P) \leq 17$ and $h(P) \leq 41$.

2. Proof of Theorem 1

The tightness of the bound 14

We start with the $(3, 4, 4, 4)$ Archimedean solid $A(3, 4, 4, 4)$, which is a 4-regular 3-polytope whose each vertex is incident with a 3-face and three 4-faces. Now cap each 4-face of $A(3, 4, 4, 4)$ to obtain a triangulation T whose each face is incident with a 4-vertex and two 7^+-vertices. (To cap a face f means to put a vertex v inside f and join v to all boundary vertices of f.) The dual D of T is a cubic 3-polytope, and we replace all its vertices by small 3-faces.

The resulting 3-polytope R is cubic and such that each vertex is incident with a 3-face, 8-face, and 14^+-face. Capping all 8^+-faces of R yields a desired 3-polytope in which every 5-vertex has a 14^+-neighbor and another 8^+-neighbor.

Discharging

Suppose that a 3-polytope P'_5 is a counterexample to the main statement of Theorem 1. Thus each minor 5-star in P'_5 contains a 15^+-vertex and at most three 5-vertices.

Let P_5 be a counterexample having the same number of vertices as P'_5 and the maximum possible number of edges.

Remark 2. P_5 has no 4^+-face with two nonconsecutive 8^+-vertices along the boundary, for otherwise adding a diagonal between these vertices would result in a counterexample with greater number of edges.

Let V, E, and F be the sets of vertices, edges, and faces of P_5. Euler's formula $|V| - |E| + |F| = 2$ implies

$$\sum_{v \in V}(d(v) - 6) + \sum_{f \in F}(2d(f) - 6) = -12. \tag{1}$$

We assign an *initial charge* $\mu(v) = d(v) - 6$ to each $v \in V$ and $\mu(f) = 2d(f) - 6$ to each $f \in F$, so that only 5-vertices have negative initial charge. Using the properties of P_5 as a counterexample to Theorem 1, we define a local redistribution of