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# Null decomposition of trees

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ABSTRACT

Let $T$ be a tree. We show that the null space of the adjacency matrix of $T$ has relevant information about the structure of $T$. We introduce the Null Decomposition of trees, which is a decomposition into two different types of trees: N-trees and S-trees. N-trees are the trees that have a unique maximum (perfect) matching. S-trees are the trees with a unique maximum independent set. We obtain formulas for the independence number and the matching number of a tree using this decomposition. We also show how the number of maximum matchings and the number of maximum independent sets in a tree are related to its null decomposition.



## 1. Introduction

The Eigenspaces of graphs have been studied for many years. The standard reference on the topic is [3]. Fiedler (1975) was the first to study graph structures with eigenvectors, see [6]. In 1988, Power used eigenvectors to study the connection structure of graphs, see [14]. The null space has been studied for many classes of graphs (paths, trees, cycles, circulant graphs, hypercubes, etc.). But, compared to the amount of research on spectral graph theory, the study of eigenvectors of graphs has received little attention.

The nullity of a tree can be given explicitly in terms of the matching number of the tree. In 2005, Fiorini, Gutman, and Sciriha, see [7], proved that among all the $n$-vertex trees whose vertex degree do not exceed a certain value $D$, the greatest nullity is $n - 2\lceil \frac{n-1}{D} \rceil$. They also gave methods for constructing trees with maximum nullity. Their work is based on the fact that for any tree $T$, $\text{null}(T) = v(T) - 2\nu(T)$, where $\nu(T)$ is the matching number of $T$. This is another consequence of the well-known fact that for trees, the characteristic and the matching polynomials are equal.

Sander and Sander (2009) worked with ideas similar to ours, see [17], but with different aims. They present a very interesting composition–decomposition technique that correlates tree eigenvectors with certain eigenvectors of an associated skeleton forest (via some contractions). They use the matching properties of a skeleton in order to determine the multiplicity of the corresponding tree eigenvalue. Their results allow them to characterize trees whose eigenspaces admit a basis consisting of vectors with entries in $\{-1, 0, 1\}$.

The purpose of this study is to determine which information about a tree can be obtained from the support of the null space of its adjacency matrix. We will introduce a new family of trees, the S-trees, which are based on the nonzero entries of vectors in the null space. We will show that every tree can be decomposed into a forest of S-trees and a forest of non-singular trees.

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Our work can be seen as a further step (in a narrow sense) of Nylen's work, see [13], and (part of) Neumaier's work (specifically, section 3 of [12]). The null decomposition of trees contradicts Theorem 3.4 (ii) and Proposition 3.6 (ii)-(v) in [12]. We give a counterexample for said theorem and proposition.

Now, some basic notation. As usual in combinatorics, $[k] := \{1, \dots, k\}$. The cardinality of a set $X$ is denoted $|X|$. We follow West's advice: the difference of two sets $A$ and $B$ will be denoted by $A - B$. Instead of $A - \{x\}$ we just write $A - x$. In this work, we will only consider finite, loopless, labeled, and simple graphs. Let $G$ be a graph, $V(G)$ is its set of vertices, and $v(G) := |V(G)|$. An $n$-graph is a graph of order $n$. On the other hand, $E(G)$ is its set of edges, and $e(G) := |E(G)|$. For all graph-theoretic notions not defined here, the reader is referred to [4]. Let $u, v \in V(G)$, with $G + \{u, v\}$ we denote the graph obtained by adding up the edge $\{u, v\}$ to $E(G)$. Let $e \in E(G)$, with $G - e$ we denote the graph obtained by removing the edge $e$ from $G$, thus $E(G - e) = E(G) - \{e\}$.

We now describe how this paper is organized. Section 2 is concerned with the notion of support of vectors associated to graphs. Section 3 defines and studies S-trees. In Section 4, we state and prove our main result: the Null Decomposition of trees. We use it in order to obtain formulas for the independence number and the matching number of a tree. We also prove that the number of maximum matchings and the number of maximum independent sets in a tree depend on its null decomposition.

## 2. Supports

We wish to investigate what information of a tree is coded in the null space of its adjacency matrix. We emphasize that it is enough to look at the nonzero coordinates of vectors in the null space of the adjacency matrix of a tree in order to obtain relevant information about the structure of every tree.

**Definition 2.1.** Let $x$ be a vector of $\mathbb{R}^n$. The **support** of $x$, denoted by $\mathcal{S}upp_{\mathbb{R}^n}(x)$, is $\{v \in [n] : x_v \neq 0\}$.

The cardinality of the support of $x$ is denoted by $\text{supp}_{\mathbb{R}^n}(x)$. Similarly, given $S \subset \mathbb{R}^n$, the **support** of $S$, denoted by $\mathcal{S}upp_{\mathbb{R}^n}(S)$, is $\bigcup_{x \in S} \mathcal{S}upp_{\mathbb{R}^n}(x)$. The cardinality of the support of $S$ is denoted by $\text{supp}_{\mathbb{R}^n}(S)$.

For example, consider the following set of vectors of $\mathbb{R}^6$:

$$S = \{(0, 1, 0, -1, 0, 0)^t, (0, 0, 1, -1, 0, 0)^t\}$$

Thus, $\mathcal{S}upp_{\mathbb{R}^6}(S) = \{2, 3, 4\}$, and $\text{supp}_{\mathbb{R}^6}(S) = 3$.

**Lemma 2.2.** *Given a subspace $S$ of $\mathbb{R}^n$. If $\mathcal{B}$ is a basis of $S$, then*

$$\mathcal{S}upp_{\mathbb{R}^n}(S) = \mathcal{S}upp_{\mathbb{R}^n}(\mathcal{B})$$

**Lemma 2.3.** *Let $W$ be a subspace of $\mathbb{R}^n$. If $S \subset \mathcal{S}upp_{\mathbb{R}^n}(W)$, then there exists $z(S) \in W$, such that $S \subset \mathcal{S}upp_{\mathbb{R}^n}(z(S))$.*

**Proof.** Let $S = \{i_1, \dots, i_h\} \subset [n]$, where $[n] = \{1, \dots, n\}$. For each $i_j \in S$ there exists a vector $x(j) \in W$ with nonzero $j$-coordinate: $x(j)_j \neq 0$. The Algorithm 1 gives a vector $z(S)$:

**Algorithm 1**

INPUT List of vectors $\{x(1), \dots, x(h)\}$.

1. $z(1) := x(1)$.
2. FOR $t = 2$ TO $h$:
   - (a) $\alpha = \max_{1 \leqslant k \leqslant n} |z(t-1)_k|$.
   - (b) $\beta = \frac{1}{2} \min\{|x(t)_k| : x(t)_k \neq 0, \text{ for } 1 \leqslant k \leqslant n\}$.
   - (c) $z(t) = \frac{1}{\alpha} z(t-1) + \frac{1}{\beta} x(t)$.

OUTPUT $z(S) = z(h)$

Clearly Algorithm 1 runs a finite number of steps: $|S|$. On the other hand, for $2 \leqslant t \leqslant h$

$$\begin{aligned} \mathcal{S}upp_{\mathbb{R}^n}(z(t)) &= \mathcal{S}upp_{\mathbb{R}^n}(z(t-1)) \cup \mathcal{S}upp_{\mathbb{R}^n}(x(t)) \\ &= \left( \bigcup_{r=1}^{t-1} \mathcal{S}upp_{\mathbb{R}^n}(x(r)) \right) \cup \mathcal{S}upp_{\mathbb{R}^n}(x(t)) \end{aligned}$$

Hence, $S \subset \bigcup_{t=1}^{h} \mathcal{S}upp_{\mathbb{R}^n}(x(t)) = \mathcal{S}upp_{\mathbb{R}^n}(z(S))$. This discussion implies that Algorithm 1 finds the vector with the requested support. ∎