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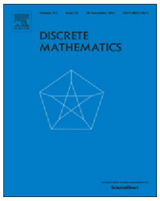

A simple characterization of special matchings in lower Bruhat intervals

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ABSTRACT

We give a simple characterization of special matchings in lower Bruhat intervals (that is, intervals starting from the identity element) of a Coxeter group. As a byproduct, we obtain some results on the action of special matchings.

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1. Introduction

Over the last few years, special matchings have shown many applications in Coxeter group theory in general (see [10,11]) and in the computation of Kazhdan–Lusztig polynomials in particular (see [2–4,12,13,15]).

The main achievements obtained by using special matchings are in the problem of the combinatorial invariance of Kazhdan–Lusztig polynomials. These famous polynomials, introduced by Kazhdan and Lusztig in [9], are polynomials indexed by pairs of elements u, v in a Coxeter group, with $u \leq v$ under Bruhat order. In the 1980s, Lusztig in private and, independently, Dyer [7] have conjectured that the Kazhdan–Lusztig polynomial $P_{u,v}(q)$ only depends on the combinatorial structure of the interval $[u, v]$ (that is, on the isomorphism class of $[u, v]$ as a poset). This conjecture is often referred to as the Combinatorial Invariance Conjecture of Kazhdan–Lusztig polynomials.

Over the last 15 years, new results about the Combinatorial Invariance Conjecture (and its generalization to the parabolic setting) for lower intervals, that is, intervals starting from the identity element, were obtained by proving a recursive formula for Kazhdan–Lusztig polynomials which depends on special matchings. This result was first obtained for the ordinary Kazhdan–Lusztig polynomials of the symmetric group [2], then for the ordinary Kazhdan–Lusztig polynomials of an arbitrary Coxeter group [4] (see also [6]), then for the parabolic Kazhdan–Lusztig polynomials of a doubly-laced Coxeter group [12], and very recently for the parabolic Kazhdan–Lusztig polynomials of an arbitrary Coxeter group [13]. Each time, these improvements were made possible by a better control on the special matchings of Coxeter groups. The impression is that a further understanding of special matchings of Coxeter groups might bring to other results on the combinatorial invariance of Kazhdan–Lusztig polynomials.

The purpose of this paper is to give a simple characterization of special matchings of lower intervals in any arbitrary Coxeter group.

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With respect to the classification given in [5], the present classification has the advantages of being more compact and simpler. It is also aesthetically pleasing to have only one self-dual type of system instead of two types of systems, one dual to the other. Furthermore, we provide a simpler description of the action of a special matching, which is easy to work with. More precisely, given a special matching M of an element $w \in W$, we show that, for all $u \in W$ with $u \leq w$, the element $M(u)$ can be computed from any factorization of u satisfying certain hypotheses, while in [5] we could compute $M(u)$ only starting from one specific factorization.

As an application of the classification in this paper, we give a new proof of the Combinatorial Invariance Conjecture for lower Bruhat intervals in any Coxeter group, which is shorter than the original one in [4].

2. Statements of main results

We fix an arbitrary Coxeter system (W, S).

For $w \in W$, the *support* of w, denoted $\mathrm{Supp}(w)$, is $\{s \in S : s \leq w\}$. For $H \subseteq S$, we let $\mathrm{Supp}_H(w)$ denote the intersection $\mathrm{Supp}(w) \cap H$. For $s \in S$, we denote by C_s the set of generators commuting with s, that is $\{c \in S : sc = cs\}$. For $w \in W$ and $H \subseteq S$, we denote by $w_0(H)$ the longest element of $[e, w] \cap W_H$.

Definition 2.1. Let $s \in S$ and $J, K \subseteq S$. We say that J and K are *s-complementary* if $J \cup K = S$ and $J \cap K = C_s$. We also say that K is the *s-complement* of J, and vice versa.

Definition 2.2. A *system* $\mathcal{S}$ for $w \in W$ is a triple (J, H, M) with $H \subseteq S$, $|H| \in \{1, 2\}$, M a matching of the Hasse diagram of $[e, w_0(H)]$, and $C_{M(e)} \subseteq J \subseteq S$, such that

[S0] M is a multiplication matching (if and) only if $|H| = 1$;
[S1] $w^J \in W_K$ (equivalently ${}^K w \in W_J$), where K is the $M(e)$-complement of J;
[S2]
- $|\mathrm{Supp}_H((w^J)^H)| \leq 1$ and if $\alpha \in \mathrm{Supp}_H((w^J)^H)$ then M commutes with λ_α,
- $|\mathrm{Supp}_H({}^H({}^K w))| \leq 1$ and if $\beta \in \mathrm{Supp}_H({}^H({}^K w))$ then M commutes with ρ_β.

Definition 2.3. Let $\mathcal{S}$ be a system (J, H, M) for w, and $u \in W$. An $\mathcal{S}$*-factorization* of u is a factorization of $u = a \cdot b \cdot c$, for some $a, b, c \in W$ satisfying the following properties:

- $\ell(u) = \ell(a) + \ell(b) + \ell(c)$;
- $a \in W_K \cap W^H$, $|\mathrm{Supp}_H(a)| \leq 1$, and if $\alpha \in \mathrm{Supp}_H(a)$ then M commutes with λ_α;
- $b \in W_H$;
- $c \in W_J \cap {}^H W$, $|\mathrm{Supp}_H(c)| \leq 1$, and if $\beta \in \mathrm{Supp}_H(c)$ then M commutes with ρ_β.

In the sequel, we prove that, to each system $\mathcal{S} = (J, H, M)$, we can attach a special matching $M_{\mathcal{S}}$ of w by letting, for all $u \in W$ with $u \leq w$

$$M_{\mathcal{S}}(u) = a \cdot M(b) \cdot c$$

where $u = a \cdot b \cdot c$ is an arbitrary $\mathcal{S}$-factorization. While it is easy to see that $\mathcal{S}$-factorizations exist for all $u \in W$ with $u \leq w$, the fact that $M_{\mathcal{S}}$ does not depend on the chosen $\mathcal{S}$-factorization and is indeed a matching of w (when $M_{\mathcal{S}}(w) \lhd w$) is a key point in the sequel.

The main results of this work are collected in the following.

Theorem. *Let (W, S) be an arbitrary Coxeter system and w be an arbitrary element of W.*

- *If M is a special matching of w, then there exists a system $\mathcal{S}$ for w such that $M = M_{\mathcal{S}}$.*
- *Vice versa, if $\mathcal{S}$ is a system for w such that $M_{\mathcal{S}}(w) \lhd w$, then $M_{\mathcal{S}}$ is a special matching for w.*
- *If M is a special matching of w associated with a system $\mathcal{S}$, $w = a \cdot b \cdot c$ is an $\mathcal{S}$-factorization of w, and u is an element smaller than w, then*

$$M(u) = a' M_{\mathcal{S}}(b') c',$$

for all factorizations $u = a' \cdot b' \cdot c'$ of u such that $a' \leq a$, $b' \leq b$, $c' \leq c$, and $\ell(u) = \ell(a') + \ell(b') + \ell(c')$.
- *Moreover, let SM_w be the set of systems $\mathcal{S} = (J, H, M)$ for w such that*
  - *$M_{\mathcal{S}}(w) \lhd w$,*
  - *for all $r \in S$,*

$$M_{\mathcal{S}}(r) = sr \text{ if and only if } r \in J.$$

  Then the set

$$\{M_{\mathcal{S}} : \mathcal{S} \in SM_w\}$$

  is a complete list of all distinct special matchings of w.

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