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Between Shi and Ish

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ABSTRACT

We introduce a new family of hyperplane arrangements in dimension $n \geq 3$ that includes both the Shi arrangement and the Ish arrangement. We prove that all the members of a given subfamily have the same number of regions – the connected components of the complement of the union of the hyperplanes – which can be *bijectively* labeled with the Pak–Stanley labeling. In addition, we show that, in the cases of the Shi and the Ish arrangements, the number of labels with *reverse centers* of a given length is equal, and conjecture that the same happens with all of the members of the family.

1. Introduction

In this paper we introduce a family of hyperplane arrangements in general dimension "between Ish and Shi", that is, formed by hyperplanes that are hyperplanes of the Shi arrangement or hyperplanes of the Ish arrangement, all of the same dimension.

More precisely, we consider, for an integer $n \geq 3$, hyperplanes of $\mathbb{R}^n$ of three different types. Let, for $1 \leq i < j \leq n$,

$$C_{ij} = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i = x_j\right\};$$
$$S_{ij} = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i = x_j + 1\right\};$$
$$I_{ij} = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 = x_j + i\right\}.$$

Note that the n-dimensional Coxeter arrangement is $\mathrm{Cox}_n = \bigcup_{1\leq i<j\leq n} C_{ij}$, the n-dimensional Shi arrangement is $\mathrm{Shi}_n = \mathrm{Cox}_n \cup \bigcup_{1\leq i<j\leq n} S_{ij}$ and the n-dimensional Ish arrangement, recently introduced by Armstrong (cf. [2]), is $\mathrm{Ish}_n = \mathrm{Cox}_n \cup \bigcup_{1\leq i<j\leq n} I_{ij}$.

Set $[n] := \{1, \ldots, n\}$ and $(k, n) = \{k+1, \ldots, n-1\}$ for $1 \leq k < n$, and define, for any $X \subseteq (1, n)$,

$$\mathcal{A}^X := \mathrm{Cox}_n \cup \bigcup_{\substack{i\in X\\ i<j\leq n}} S_{ij} \cup \bigcup_{\substack{i\in[n]\setminus X\\ i<j\leq n}} I_{ij},$$

so that $\mathrm{Shi}_n = \mathcal{A}^{(1,n)}$ and $\mathrm{Ish}_n = \mathcal{A}^{\varnothing} = \mathcal{A}^{(n-1,n)}$.

We study the arrangements of form $\mathcal{A}^X$ for $X \subseteq (1, n)$, with special interest in the Pak–Stanley labelings of the regions of the arrangements. The labels are $\mathcal{G}$-*parking functions* for special directed multi-graphs $\mathcal{G}$ as defined by Mazin [9]. In particular, we show that, in the case where $X = (k, n)$ for some $1 \leq k < n$, there are $(n+1)^{n-1}$ regions which are *bijectively* labeled.

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The notion of G-parking function was introduced by Postnikov and Shapiro in the construction of two algebras related to a general *undirected* graph G [11]. Later, Hopkins and Perkinson [7] showed that the labels of the Pak–Stanley labeling of the regions of a given hyperplane arrangement defined by G are exactly the G-parking functions, a fact that had been conjectured by Duval, Klivans and Martin [6]. Recently, Mazin [9] generalized this result to a very general class of hyperplane arrangements, with a similar concept based on a general *directed multi*-graph $\mathcal{G}$. Whereas Hopkins and Perkinson's hyperplane arrangements include for example the (original) multidimensional Shi arrangement, Mazin's hyperplane arrangements include the multidimensional k-Shi arrangement, the multidimensional Ish arrangement and in fact all the arrangements we consider here.

We start this study in Section 2 by showing (cf. Theorem 2.2) that all arrangements of form $\mathcal{A}^{(k,n)}$ $(1 \le k < n)$ have the same *characteristic polynomial*, namely

$$\chi(q) = q(q-n)^{n-1}\,,$$

from which it follows that they all have the same number of regions as well as the same number of relatively bounded regions, namely $(n+1)^{n-1}$ and $(n-1)^{n-1}$, by a famous result of Zaslavsky [15].

Since, by a result of Mazin [9, Theorem 3.1.] the corresponding labels are exactly the $\mathcal{G}$-parking functions, by showing that they are also $(n+1)^{n-1}$ we prove (cf. Theorem 3.7) that the labelings are bijective.

For this purpose, in Section 3 we extend to directed multi-graphs a result of Postnikov and Shapiro [11, Theorem 2.1.] which says (cf. Proposition 3.5) that the number of G-parking functions is the number of spanning trees of G. More precisely, we extend Perkinson, Yang and Yu's depth-first search version of Dhar's burning algorithm [10], which provides an explicit bijection between both sets, to directed multi-graphs. We also obtain from this algorithm (cf. Proposition 3.12) a characterization of the labels of the Ish_n arrangement (the *Ish-parking functions* of dimension n).

In Section 4 we show that the number of parking functions with *reverse center* (which is essentially the *center* [5] of the reverse word — see Definition 3.10) of a given length is exactly the number of Ish-parking functions with reverse center of the same length. This result follows in the direction of a previous paper of ours [4], on which it is based, and paves the way to Conjecture 4.4, which says that the same happens when the Ish-parking functions are replaced by the $\mathcal{G}$-parking functions corresponding to the new arrangements introduced here. However, opposite to what happens with the Shi and the Ish arrangements (and, in general, with the arrangements of form $\mathcal{A}^{(k,n)}$), the number of $\mathcal{G}$-parking functions can be less than the number of regions. In this paper, the Pak–Stanley labeling used in the Shi arrangement is a variation of the original definition by Pak and Stanley. In an Appendix, we explain how to use for this variation the construction developed in another paper of ours [5] for inverting the original Pak–Stanley labeling.

2. The characteristic polynomial

In the following, we evaluate the *characteristic polynomial* $\chi(\mathcal{A}^X, q)$ in the case where $X = (k, n)$. We show that, in this particular case, $\chi(\mathcal{A}^{(k,n)}, q) = \chi(\mathrm{Shi}_n, q) = \chi(\mathrm{Ish}_n, q)$. In other words, the characteristic polynomial of $\mathcal{A}^{(k,n)}$ does not depend on $1 \le k < n$.

For this purpose, we use the finite field method [1,3], which means that we evaluate $\chi(\mathcal{A}^{(k,n)}, q)$ for any sufficiently large prime number q by counting the number of elements of the set X_q formed by the elements $(x_1, \ldots, x_n) \in \mathbb{F}_q^n$ that verify, for each $1 \le i < j \le n$, the conditions

$$x_i \neq x_j \wedge \begin{cases} x_1 \neq x_j + i, & \text{if } i \le k; \\ x_i \neq x_j + 1, & \text{if } i > k. \end{cases}$$

For convenience sake, fixed n, we see the elements of

$$S_q^n = \left\{(x_1, \ldots, x_n) \in \mathbb{F}_q^n \mid \forall 1 \le i < j \le n\,,\ x_i \neq x_j\right\}$$

as injective labelings in $\mathbb{F}_q$ of the elements of $[n]$ [1,14]. For instance, for $n = 5$ and $q = 13$, the labeling represented below corresponds to $\mathbf{x} := (2, 9, 3, 11, 12) \in S_{13}^5$. Note that if $(x_1, \ldots, x_5) = \mathbf{x}$ then $x_1 = x_5 + 3$. Hence, $\mathbf{x} \in I_{3,5}$ and so $\mathbf{x} \notin X_{13}$ for $X = \varnothing$. On the other hand, for no $1 \le i < j \le 5$ is $x_i = x_j + 1$, and so $\mathbf{x} \in X_{13}$ for $X = (1, 5)$.