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On a class of quaternary complex Hadamard matrices

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ABSTRACT

We introduce a class of regular unit Hadamard matrices whose entries consist of two complex numbers and their conjugates for a total of four complex numbers. We then show that these matrices are contained in the Bose–Mesner algebra of an association scheme arising from skew Paley matrices.

1. Introduction

An $n \times n$ matrix H is a *unit Hadamard matrix* if its entries are all complex numbers of modulus 1 and it satisfies $HH^* = nI_n$. If the entries of H are all complex kth roots of unity, it is called a *Butson Hadamard matrix*, referred to as a BH(n, k), and the particular case of $k = 2$ is a *Hadamard matrix*. Following Compton et al. [4] we call a Butson or unit Hadamard matrix *unreal* if its entries are strictly in $\mathbb{C} \setminus \mathbb{R}$. A Hadamard matrix H of order n is said to be of *skew type* if $H = I + W$, where W is a skew symmetric $(0, \pm1)$-matrix. It follows that $WW^T = (n-1)I_n$. For a thorough examination of unit and Butson Hadamard matrices, we refer the reader to Szöllősi's Ph.D. thesis [14], and for some fundamental results and applications of Hadamard matrices, we refer the reader to Seberry and Yamada's 1992 survey [13]. Given a matrix A of order n, let R_i denote the ith row of A, $S(R_i)$ the sum of all entries of R_i, and $S(A)$, called the *excess of* A, the sum of all its entries. A result of [1] implies that for a unit Hadamard matrix of order n, $|S(A)| \leq n\sqrt{n}$ and equality occurs if and only if $|S(R_i)| = \sqrt{n}$ for $1 \leq i \leq n$. A unit Hadamard matrix A of order n is called *regular* if $|S(R_i)| = \sqrt{n}$ for $1 \leq i \leq n$, see [10] for details.

In this paper we introduce a recursive method to construct pairs of (±1)-matrices satisfying two specific equations. Similar recursive methods were presented in 2005 to obtain symmetric designs and orthogonal designs [5,9].

Assuming the existence of a skew type Hadamard matrix of order $q+1$, we show the pairs of matrices obtained from our recursive method can be used to construct infinite classes of a special type of unit Hadamard matrices of order q^m, for each positive integer m, which we have dubbed *quaternary unit Hadamard* matrices. In particular, we conclude that for each prime power $q \equiv 3 \pmod 4$ and integer $m \geq 0$, there is an unreal quaternary unit Hadamard matrix of order q^m and an unreal BH$(3^m, 6)$. Moreover, we will demonstrate that all of the constructed Butson Hadamard matrices and quaternary unit Hadamard matrices are regular, and some of those have multicirculant structure.

Some of the results in this paper are closely related to part of the results in a recent paper by Compton et al. [4], see also [11]. Among other results, Compton et al. proved the existence of BH$(3^m, 6)$s for each integer $m \geq 0$. Herein, we too will construct BH$(3^m, 6)$s. However, our matrices are distinguished from those of Compton et al. in that our BH$(3^m, 6)$s are regular and multicirculant. In their paper, Compton et al. also showed that a BH$(n, 6)$ is equivalent to a pair of amicable (±1)-matrices satisfying a certain equation, and that this pair of matrices can be used to construct a Hadamard matrix. We have generalized this result in Section 2 by introducing quaternary unit Hadamard matrices and showing that they are equivalent to a pair of amicable (±1)-matrices satisfying an equation analogous to that introduced by Compton et al. Next, in Section 3 we will introduce a recursive method to construct such pairs of matrices, and we will use this method to show

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that for each prime power $q \equiv 3 \pmod 4$ and integer $m \geq 0$, we can construct an unreal quaternary unit Hadamard matrix of order q^m and an unreal BH$(3^m, 6)$. Finally, in Chapter 4, we introduce an association scheme whose Bose–Mesner algebra contains our quaternary unit Hadamard matrices.

2. Quaternary unit Hadamard matrices

Definition 2.1. We say that an $n \times n$ unit Hadamard matrix H is *quaternary* if there is a positive integer m such that the entries of H are all in the set $\left\{\pm\frac{1}{\sqrt{m+1}} \pm i\sqrt{\frac{m}{m+1}}, \pm\frac{1}{\sqrt{m+1}} \mp i\sqrt{\frac{m}{m+1}}\right\}$. For short, we refer to such a quaternary unit Hadamard matrix as a QUH(n, m).

It is readily verified that any QUH$(n, 1)$ or QUH$(n, 3)$ is also a Butson Hadamard matrix.

Lemma 2.2. *Let m be a positive integer. Then $\zeta = \frac{1}{\sqrt{m+1}} + i\sqrt{\frac{m}{m+1}}$ is a root of unity if and only if $m = 1$ or $m = 3$.*

Proof. If ζ is a root of unity, then so are ζ^2 and $\overline{\zeta}^2$. Thus $\zeta^2 + \overline{\zeta}^2 = \frac{-2(m-1)}{m+1}$ is an algebraic integer, and hence an integer. This implies that $m = 1$ or 3. □

The next proposition follows immediately from the previous lemma and the observation that any QUH$(n, 1)$ or QUH$(n, 3)$ is also a Butson Hadamard matrix.

Proposition 2.3. *A QUH(n, m) is a Butson Hadamard matrix if and only if $m = 1$ or $m = 3$.*

We now demonstrate that QUH(n, m)s are equivalent to pairs of $n \times n$ matrices satisfying certain properties. First, however, recall a definition.

Definition 2.4. Two complex matrices A and B are called *amicable* if $AB^* = BA^*$.

In the Ref. [4], Compton et al. establish the following result.

Theorem 2.5 (*Compton et al., [4]*). *An unreal BH$(n, 6)$ is equivalent to a pair of $n \times n$ amicable (± 1)-matrices A and B satisfying $AA^T + 3BB^T = 4nI_n$.*

With little difficulty, this result can be generalized in the following manner. Assume H is a QUH(n, m). Then we can write

$$H = \frac{1}{\sqrt{m+1}}A + i\sqrt{\frac{m}{m+1}}B$$

for some (± 1)-matrices A and B. Therefore,

$$nI_n = \left(\frac{1}{\sqrt{m+1}}A + i\sqrt{\frac{m}{m+1}}B\right)\left(\frac{1}{\sqrt{m+1}}A + i\sqrt{\frac{m}{m+1}}B\right)^*,$$

so

$$n(m+1)I_n = AA^T + mBB^T + i\sqrt{m}(BA^T - AB^T).$$

Since the right-hand-side must be real, this proves the following generalization of Theorem 2.5.

Theorem 2.6. *A QUH(n, m) is equivalent to a pair of $n \times n$ amicable (± 1)-matrices A and B satisfying $AA^T + mBB^T = (m+1)nI_n$.*

3. A recursive method

In this section we will introduce a recursive construction for pairs of matrices satisfying the aforementioned properties. We use j_n and J_n to denote the $1 \times n$ and $n \times n$ all-ones matrices respectively. Subscripts will be dropped where no ambiguity arises.

Let $q + 1$ be the order of a skew type Hadamard matrix H. Multiply rows and columns of H, if necessary, to get the matrix

$$\begin{pmatrix} 1 & j \\ -j^T & I + Q \end{pmatrix}.$$

The $(0, \pm 1)$-matrix $Q = (q_{ij})_{i,j=1}^q$, called the *skew symmetric core* of the skew type Hadamard matrix, is skew symmetric, $J_q Q = QJ_q = 0$, and $QQ^T = qI_q - J_q$. For any odd prime power q the Jacobsthal matrix of order q defined by

$$q_{ij} = \chi_q(a_i - a_j)\,(a_i, a_j \in GF(q))$$