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Enumerating cycles in the graph of overlapping permutations

John Asplund [a], N. Bradley Fox [b,*]

[a] Department of Technology and Mathematics, Dalton State College, Dalton, GA 30720, USA
[b] Department of Mathematics and Statistics, Austin Peay State University, Clarksville, TN 37044, USA

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ABSTRACT

The graph of overlapping permutations is a directed graph that is an analogue to the De Bruijn graph. It consists of vertices that are permutations of length n and edges that are permutations of length $n+1$ in which an edge $a_1 \cdots a_{n+1}$ would connect the standardization of $a_1 \cdots a_n$ to the standardization of $a_2 \cdots a_{n+1}$. We examine properties of this graph to determine where directed cycles can exist, to count the number of directed 2-cycles within the graph, and to enumerate the vertices that are contained within closed walks and directed cycles of more general lengths.

1. Introduction

In this paper we will discuss an analogue to a classical object in combinatorics, De Bruijn graphs. The graphs in this paper are all directed graphs with arcs (edges with an orientation) between vertices. For a set $\{0, 1, \dots, q-1\}$, let $\{0, 1, \dots, q-1\}^n$ be the set of all strings of length n with elements from $\{0, 1, \dots, q-1\}$. A De Bruijn graph has the vertex set $\{0, 1, \dots, q-1\}^n$ and a directed edge from each vertex $x_1x_2 \cdots x_n$ to the vertex $x_2x_3 \cdots x_{n+1}$. In other words, there is an edge from $\mathbf{a}$ to $\mathbf{b}$ if and only if the last $n-1$ elements of $\mathbf{a}$ and the first $n-1$ elements of $\mathbf{b}$ are the same. One of the properties of De Bruijn graphs with vertex set $\{0, 1, \dots, q-1\}^n$ that has been studied is the number of directed cycles of length k, for $k \leq n$, which is shown in [4]. For ease of notation, we will call a directed cycle of length k a k-cycle.

We will examine the number of k-cycles in the graph of overlapping permutations, $G(n)$, which was introduced in [2] and was studied in [1] and Chapter 5 of [7]. The existence of overlapping cycles on a similar graph can also be seen in [5]. For a permutation $\mathbf{a} = a_1a_2 \cdots a_n$, we denote the *standardization* of the substring $a_{s+1}a_{s+2} \cdots a_{s+t}$ by $\mathrm{st}(a_{s+1}a_{s+2} \cdots a_{s+t}) = b_1b_2 \cdots b_t$ where $b_i \in \{1, \dots, t\}$ with $b_i < b_j$ if and only if $a_{s+i} < a_{s+j}$ for all $1 \leq i, j \leq t$. Note that applying the standardization of a length t substring results in a unique permutation in the symmetric group $\mathfrak{S}_t$. Furthermore, $\mathrm{st}(a_{s+1}a_{s+2} \cdots a_{s+t})$ is the permutation that is *order isomorphic* to $a_{s+1}a_{s+2} \cdots a_{s+t}$, using the concept developed by Johnson [6] to describe two words of length n for which the relative order of their elements is the same.

Let $G(n)$ be the graph with vertex set as the set of all permutations of length n and consisting of a directed edge from a vertex $\mathbf{a} = a_1a_2 \cdots a_n$ to $\mathbf{b} = b_1b_2 \cdots b_n$ for each permutation $\pi = \pi_1\pi_2 \cdots \pi_n\pi_{n+1} \in \mathfrak{S}_{n+1}$ with $\mathrm{st}(\pi_1\pi_2 \cdots \pi_n) = \mathbf{a}$ and $\mathrm{st}(\pi_2\pi_3 \cdots \pi_{n+1}) = \mathbf{b}$. The existence of an edge from $\mathbf{a}$ to $\mathbf{b}$ directly implies that $\mathrm{st}(a_2a_3 \cdots a_n) = \mathrm{st}(b_1b_2 \cdots b_{n-1})$. This adjacency condition is why this is called a graph of overlapping permutations and is analogous to the overlapping condition of strings in the De Bruijn graph. Be aware that $G(n)$ can have multiple directed edges between the same vertices and in the same direction.

Enumerating the cycles in the graph of overlapping permutations was first attempted by Ehrenborg, Kitaev, and Steingrímsson in [3]. The authors focused on the graph $G(n, 312)$, which is the graph of overlapping permutations where the

* Corresponding author.
E-mail addresses: jasplund@daltonstate.edu (J. Asplund), foxb@apsu.edu (N.B. Fox).

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vertices and edges are permutations which avoid the pattern 312, i.e., a subsequence of three entries in the permutations of the vertices and edges whose standardization is 312. They determined the number of closed walks of length k for $k \leq n$, as well as the number of k-cycles. See Theorems 5.1 and 5.2 in [3] for these results. An important observation for these enumerations is that they do not depend upon n. This is not the case for $G(n)$, as increasing the length of the permutations will increase the number of k-cycles when looking at the entire graph $G(n)$ instead of a subgraph that avoids certain patterns.

In the following section, we will discuss some pertinent graph theory terminology and present an example of $G(3)$. In Section 3, we will introduce conditions on which a closed walk exists at a particular vertex, as well as discuss the existence of two edges between the same two vertices and multiple closed walks stemming from a single vertex. In Section 4, we count the 2-cycles in $G(n)$. Section 5 includes enumerations for the number of vertices contained within closed walks and a correspondence between the number of closed k-walks in $G(n)$ and the number of k-cycles in $G(n)$ as long as k is prime. Finally, Section 6 consists of further research questions involving cycles within the graph of overlapping permutations.

2. Preliminaries and an example

Before we introduce our results, we must first establish some graph theory terminology. The following definitions are for directed graphs. We consider a sequence $(v_1, e_1, v_2, e_2, v_3, \ldots, v_k, e_k, v_{k+1})$ of vertices v_i and edges $e_i = (v_i, v_{i+1})$ to be a *walk* of length k, or a k-walk. We will only list the vertices in the walk for simplification within our proofs, although we do consider two walks (and two cycles) to be distinct if the sequence of vertices are the same, but edges differ due to pairs of vertices being connected by multiple edges. If $v_1 = v_{k+1}$, the walk is called a *closed walk* and will usually be written as the sequence $(v_1, v_2, \ldots, v_k)$ to represent the closed k-walk. If the vertices are all distinct, the closed k-walk is a *k-cycle*. For more information about graph terminology not mentioned in this article, see [8].

When counting closed walks, we consider the closed walk $(v_1, e_1, v_2, e_2, v_3, \ldots, v_k, e_k, v_1)$ to be the same as the closed walk $(v_j, e_j, v_{j+1}, e_{j+1}, v_{j+2}, \ldots, v_k, e_k, v_1, e_1, v_2, \ldots, v_{j-1}, e_{j-1}, v_j)$ that has a different starting vertex. Likewise, they are counted as one cycle if the vertices are distinct. More formally, we are attempting to count equivalence classes of closed walks (and cycles) by considering shifted sequences of vertices and edges within a closed walk to be equivalent.

Given a vertex $\mathbf{a} = a_1a_2\cdots a_n$, we consider two important vertices related to it. First, the *complement* of $\mathbf{a}$ is the vertex $\bar{\mathbf{a}} = (n+1-a_1)(n+1-a_2)\cdots(n+1-a_n)$. Second, we define the *one position cyclic shift* of the vertex $\mathbf{a}$ by $\sigma(\mathbf{a}) = a_2a_3\cdots a_na_1$. We will simply refer to this as a cyclic shift from this point on, although we strictly mean one position within the permutation is being shifted.

Fig. 1 displays the graph $G(3)$ with permutations of length 3 as the six vertices. The twenty-four edges, although unlabeled in the figure, correspond to the permutations of length 4. Recall the definition of the edges in $G(n)$ as $c_1\cdots c_{n+1}$ connecting the standardization $\mathbf{a} = a_1\cdots a_n = \text{st}(c_1\cdots c_n)$ to the standardization $\mathbf{b} = b_1\cdots b_n = \text{st}(c_2\cdots c_{n+1})$. This directly implies there is an edge from $\mathbf{a}$ to $\mathbf{b}$ if and only if $\text{st}(a_2\cdots a_n) = \text{st}(b_1\cdots b_{n-1})$. Hence we see in this example that there exists an edge from 123 to 132 since $\text{st}(23) = \text{st}(13) = 12$, whereas there is no returning edge from 132 to 123 since $\text{st}(32) = 21 \neq \text{st}(12) = 12$.

The number of cycles in the graph is as follows: two 1-cycles at the trivial vertices 123 and 321, six 2-cycles, and twenty-six 3-cycles. Observe that with 2-cycles some pairs of vertices create two 2-cycles because there are multiple edges between the same two vertices. An example is the cycle (132, 213) made from either the edge pair 1324 and 2143 or the pair 1324 and 3142. As many as eight 3-cycles can be made from a triple of vertices, which occurs with the 3-cycle (132, 321, 213), which can make the enumeration of long cycles rather difficult. Additionally, note that vertices can be included in multiple cycles with differing vertices. For instance, the vertex 231 is contained in the 2-cycles (231, 312) and (231, 213). A final observation is that the trivial vertices are within k-cycles for each $1 \leq k \leq 6$ except for $k = 2$. The fact that they are not in 2-cycles will be generalized in the next section.

3. Properties of closed walks and cycles in $G(n)$

We will now introduce properties regarding the inclusion of vertices within closed walks and cycles. Unless otherwise noted, assume k is an integer with $2 \leq k \leq n-1$. We first establish a necessary condition for the existence of a closed k-walk through a vertex.

Theorem 1. *If a vertex $\mathbf{a} = a_1a_2\cdots a_n$ is in some closed k-walk in $G(n)$, then*

$$st(a_1a_2\cdots a_{n-k}) = st(a_{k+1}a_{k+2}\cdots a_n).$$

Proof. Assume $\mathbf{a} = a_1a_2\cdots a_n$ is a vertex in the closed k-walk $\left(\mathbf{a}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(k)}\right)$ where $\mathbf{a}^{(i)} = a_1^{(i)}a_2^{(i)}\cdots a_n^{(i)}$. The directed edge from $\mathbf{a}$ to $\mathbf{a}^{(2)}$ in the walk above gives us the equality $\text{st}(a_2\cdots a_n) = \text{st}\left(a_1^{(2)}\cdots a_{n-1}^{(2)}\right)$, which when narrowing down to the last $n-k$ elements of $\mathbf{a}$ becomes

$$\text{st}(a_{k+1}\cdots a_n) = \text{st}\left(a_k^{(2)}\cdots a_{n-1}^{(2)}\right).$$

Then we use the second directed edge in this walk, which provides the equality $\text{st}\left(a_2^{(2)}\cdots a_n^{(2)}\right) = \text{st}\left(a_1^{(3)}\cdots a_{n-1}^{(3)}\right)$. By combining this with the previous equality we attain

$$\text{st}(a_{k+1}\cdots a_n) = \text{st}\left(a_{k-1}^{(3)}\cdots a_{n-2}^{(3)}\right).$$