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Game of cops and robbers in oriented quotients of the integer grid

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ABSTRACT

The integer grid $\mathbb{Z}\Box\mathbb{Z}$ has four typical orientations of its edges which make it a vertex-transitive digraph. In this paper we analyze the game of Cops and Robbers on arbitrary finite quotients of these directed grids.

1. Introduction

The main goal of this paper is to show that the game of Cops and Robbers in directed graphs (digraphs) can be as natural and as inspiring as the game on undirected graphs. It is known (see [3]) that the cop number (see Section 1.2 for definitions) of any (connected) planar graph is at most 3. It is also known that it is at most 4 for every graph embedded in the torus (and is currently not known whether four cops are ever needed). More generally, the cop number is bounded for graphs of bounded genus and, in fact, is bounded in any proper minor-closed family of graphs (see [2]). A natural question arises:

Problem 1.1. Is the cop number on planar Eulerian digraphs bounded or unbounded.

The same question can be asked for Eulerian digraphs of bounded genus (Here we restrict our attention to Eulerian digraphs since the game on any undirected graph G is equivalent to the game on the Eulerian digraph obtained from G by replacing each edge with a pair of oppositely oriented arcs joining the same pair of vertices.). While the main tool (that of "guarding a geodesic path", see [3]) used for undirected graphs is no longer available for digraphs, there is some hope for Problem 1.1 to have positive answer. As we show in this paper, the game can be analyzed on arbitrary 4-regular quadrangulations of the torus and the Klein bottle, at least when some "regularity" about orientations of the edges is assumed. In all treated cases, the cop number is at most 4 (see Theorems 2.1, 2.2, and 3.1) and four cops are necessary for one kind of orientation (Theorem 3.1).

In a forthcoming work [4] we show another striking fact that the cop number of arbitrary "straight-ahead orientations" of 4-regular quadrangulations is bounded (the proved upper bound is 404). This provides evidence that study of the game in (Eulerian) digraphs is of sufficient significance.

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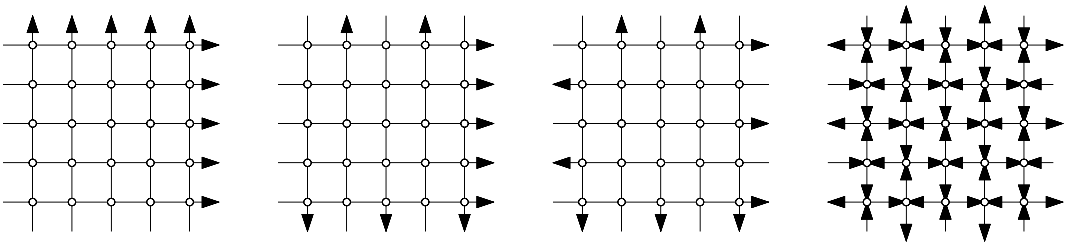
(a) Vertex of type 1. (b) Vertex of type 2.

Fig. 1. Type 1 and type 2 orientation around a vertex.



Type (1a) Type (1b) Type (1c) Type (2)

Fig. 2. Vertex-transitive orientations of the integer grid.

1.1. *Quotients of vertex-transitive orientations of the integer grid*

Consider a 4-regular quadrangulation of a surface. It follows by Euler's formula that the surface is either the torus or the Klein bottle and it can be shown by using the Gauss–Bonnet Theorem that the SAW (straight-ahead walks) partition the edges into cycles, all of which are noncontractible on the surface. These cycles can be split into two classes, each class consisting of pairwise disjoint cycles (we call them *vertical cycles* and *horizontal cycles*, respectively) such that each vertical and each horizontal cycle intersects (possibly more than once). By giving each of these cycles an orientation, we obtain an Eulerian digraph in which, at each vertex, the two incoming edges and two outgoing edges are consecutive in the local rotation around the vertex. This kind of orienting the edges is said to be of *type* (1). We will also consider *type* (2) orientations, where at each vertex, the two incoming and the two outgoing edges are opposite to each other in the local rotation. See Fig. 1. Under this orientation, each facial quadrangle is a directed 4-cycle.

The universal cover of a 4-regular quadrangulation is the 4-regular tessellation of the plane with square faces (the integer grid), and every finite quotient of the integer grid is a 4-regular quadrangulation of the torus or the Klein bottle. An orientation of the edges of such quadrangulations is said to be *special* if its lift to the universal cover gives a vertex-transitive digraph. It is not hard to see that this means one of the cases shown in Fig. 2. They are classified being of *type* (1) (subtypes (1a), (1b), (1c)) or (2), as indicated in the figure.

Four-regular quadrangulations of the torus admit a simple description. Each such quadrangulation is of the form $Q(r,s,t)$, where r,s,t are arbitrary positive integers, $0 \le t < r$, and $Q(r,s,t)$ is obtained from the $(r+1)\times(s+1)$ grid with underlying graph $P_{r+1}\,\square\, P_{s+1}$ (the cartesian product of paths on $r+1$ and $s+1$ vertices) by identifying the "leftmost" path of length s with the "rightmost" one (to obtain a cylinder) and identifying the bottom r-cycle of this cylinder with the top one after rotating the top clockwise for t edges. More precisely, the left-most bottom vertex of the path P_{r+1} is identified with the tth vertex in the top-side path; See Fig. 3. In other words, the quadrangulation $Q(r,s,t)$ is the quotient of the integer grid $\mathbb{Z}\,\square\,\mathbb{Z}$ determined by the equivalence relation generated by all pairs $(x,y) \sim (x+r,y)$ and $(x,y) \sim (x+t,y+s)$. See Fig. 3. This classification can be derived by considering appropriate fundamental polygon of the universal cover (which is isomorphic to the tessellation of the plane with squares). In graph theory, this was observed by Altschuler [1]; several later works do the same (e.g. [8]). Quadrangulations of the Klein bottle are a bit more complicated (see [6–8], or [5]). While all toroidal quadrangulations $Q(r,s,t)$ are vertex-transitive maps, this is no longer true for the Klein bottle. For our purpose it will suffice to know that the orientable double cover of such a quadrangulation Q is of the form $Q(r,s,t)$ and since it is a double cover it has $|V(Q)| = \frac{1}{2}rs$.

1.2. *Game of Cops and Robbers on digraphs*

The game of Cops and Robbers is a well studied game on finite graphs with a variety of interesting results and applications and several outstanding open questions. We refer to Bonato and Nowakowski [3] for an overview of the theory and applications behind this game.

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