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Structure and algorithms for (cap, even hole)-free graphs

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ABSTRACT

A graph is even-hole-free if it has no induced even cycles of length 4 or more. A cap is a cycle of length at least 5 with exactly one chord and that chord creates a triangle with the cycle. In this paper, we consider (cap, even hole)-free graphs, and more generally, (cap, 4-hole)-free odd-signable graphs. We give an explicit construction of these graphs. We prove that every such graph G has a vertex of degree at most $\frac{3}{2}\omega(G) - 1$, and hence $\chi(G) \leq \frac{3}{2}\omega(G)$, where $\omega(G)$ denotes the size of a largest clique in G and $\chi(G)$ denotes the chromatic number of G. We give an $O(nm)$ algorithm for q-coloring these graphs for fixed q and an $O(nm)$ algorithm for maximum weight stable set, where n is the number of vertices and m is the number of edges of the input graph. We also give a polynomial-time algorithm for minimum coloring.

Our algorithms are based on our results that triangle-free odd-signable graphs have treewidth at most 5 and thus have clique-width at most 48, and that (cap, 4-hole)-free odd-signable graphs G without clique cutsets have treewidth at most $6\omega(G) - 1$ and clique-width at most 48.

1. Introduction

In this paper all graphs are finite and simple. We say that a graph G *contains* a graph F, if F is isomorphic to an induced subgraph of G. A graph G is F-*free* if it does not contain F, and for a family of graphs $\mathcal{F}$, G is $\mathcal{F}$-*free* if G is F-free for every $F \in \mathcal{F}$. A *hole* is a chordless cycle of length at least four. A hole is *even* (respectively, *odd*) if it has an even (respectively, odd) number of vertices. A *cap* is a graph that consists of a hole H and a vertex x that has exactly two neighbors in H, that are furthermore adjacent. The graph C_n is a hole of length n, and is also called an n-*hole*. In this paper we study the class of (cap, even hole)-free graphs, and more generally the class of (cap, 4-hole)-free odd-signable graphs, which we define later.

Let G be a graph. We use n to denote the number of vertices of G and m the number of edges of G. A set $S \subseteq V(G)$ is a *clique* of G if all pairs of vertices of S are adjacent. The size of a largest clique in a graph G is denoted by $\omega(G)$, and is sometimes called the *clique number* of G. We say that G is a *complete graph* if $V(G)$ is a clique. We denote by K_n the complete graph on n vertices. The graph K_3 is also called a *triangle*. A set $S \subseteq V(G)$ is a *stable set* of G if no two vertices of S are adjacent. The size of a largest stable set of G is denoted by $\alpha(G)$. A q-*coloring* of G is a function $c : V(G) \longrightarrow \{1, \ldots, q\}$, such that $c(u) \neq c(v)$ for every edge uv of G. The *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimum number q for which there exists a q-coloring of G.

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The class of (cap, odd hole)-free graphs has been studied extensively in literature. This is precisely the class of Meyniel graphs, where a graph G is *Meyniel* if every odd length cycle of G, that is not a triangle, has at least two chords. These graphs were proven to be perfect by Meyniel [30] and Markosyan and Karapetyan [28]. Burlet and Fonlupt [5] obtained the first polynomial-time recognition algorithm for Meyniel graphs, by decomposing these graphs with amalgams (that they introduced in the same paper). Subsequently, Roussel and Rusu [33] obtained a faster algorithm for recognizing Meyniel graphs (of complexity $O(m^2)$), that is not decomposition-based. Hertz [23] gave an $O(nm)$ algorithm for coloring and obtaining a largest clique of a Meyniel graph. This algorithm is based on contractions of even pairs. It is an improvement on the $O(n^8)$ algorithm of Hoàng [24]. Roussel and Rusu [34] gave an $O(n^2)$ algorithm that colors a Meyniel graph without using even pairs. This algorithm "simulates" even pair contractions and it is based on lexicographic breadth-first search and greedy sequential coloring.

Algorithms have also been given which find a minimum coloring of a Meyniel graph, but do not require that the input graph be known to be Meyniel. A *Meyniel obstruction* is an induced subgraph which is an odd cycle with at most one chord. A *strong stable set* in a graph G is a stable set which intersects every (inclusion-wise) maximal clique of G. Cameron and Edmonds [7] gave an $O(n^2)$ algorithm which for any graph, finds either a strong stable set or a Meyniel obstruction. This algorithm can be applied at most n times to find, in any graph, either a clique and coloring of the same size or a Meyniel obstruction. Cameron, Lévêque and Maffray [8] showed that a variant of the Roussel–Rusu coloring algorithm for Meyniel graphs [34] can be enhanced to find, for any input graph, either a clique and coloring of the same size or a Meyniel obstruction. The worst-case complexity of the algorithm is still $O(n^2)$.

In [11], Conforti, Cornuéjols, Kapoor and Vušković generalize Burlet and Fonlupt's decomposition theorem for Meyniel graphs [5] to the decomposition by amalgams of all cap-free graphs. This theorem is the basis for polynomial-time recognition algorithms for cap-free odd-signable graphs and (cap, even hole)-free graphs. Since triangle-free graphs are cap-free, it follows that the problems of coloring and of finding the size of a largest stable set are both NP-hard for cap-free graphs. In [14], Conforti, Gerards and Pashkovich show how to obtain a polynomial-time algorithm for solving the maximum weight stable set problem on any class of graphs that is decomposable by amalgams into basic graphs for which one can solve the maximum weight stable set problem in polynomial time. This leads to the first known non-polyhedral algorithm for the maximum weight stable set problem for Meyniel graphs. Furthermore, using the decomposition theorems from [11] and [12], they obtain a polynomial-time algorithm for solving the maximum weight stable set problem for (cap, even hole)-free graphs (and more generally cap-free odd-signable graphs). For a survey on even-hole-free graphs and odd-signable graphs, see [40].

Aboulker, Charbit, Trotignon and Vušković [1] gave an $O(nm)$-time algorithm whose input is a weighted graph G and whose output is a maximum weighted clique of G or a certificate proving that G is not 4-hole-free odd-signable. (The crux of this algorithm was actually developed by da Silva and Vušković in [20].)

In Section 3, we give an explicit construction of (cap, 4-hole)-free odd-signable graphs, based on [11] and [12]. From this, in Section 4, we derive that every such graph G has a vertex of degree at most $\frac{3}{2}\omega(G) - 1$, and hence $\chi(G) \leq \frac{3}{2}\omega(G)$. It follows that G can be colored with at most $\frac{3}{2}\omega(G)$ colors using the greedy coloring algorithm.

In Section 5, we prove that triangle-free odd-signable graphs have treewidth at most 5 and thus have clique-width at most 48 [15]. We also prove that (cap, 4-hole)-free odd-signable graphs G without clique cutsets have clique-width at most 48 and treewidth at most $6\omega(G) - 1$.

In Section 6, we give an $O(nm)$ algorithm for q-coloring (cap, 4-hole)-free odd-signable graphs. We give a (first known) polynomial-time algorithm for finding a minimum coloring of these graphs (chromatic number). We also obtain an $O(nm)$ algorithm for the maximum weight stable set problem for (cap, 4-hole)-free odd-signable graphs. We observe that the algorithm in [14] proceeds by first decomposing the graph by amalgams, a step that takes $O(n^4m)$ time ($O(n^2m)$ to find an amalgam [16], which is called $O(n^2)$ times) and creates $O(n^2)$ blocks. For each block, $O(n)$ maximum weight stable set problems must be solved, each of which can be done in $O(n + m)$ time. Thus the overall complexity of their algorithm is $O(n^4m)$. Finally, we observe that all our algorithms are robust in the sense that we do not need to assume that the input graph is (cap, 4-hole)-free odd-signable.

It is known that planar even-hole-free graphs have treewidth at most 49 [35]. We observe that (cap, even hole)-free graphs are not necessarily planar. It is not hard to check that the graph in Fig. 1 is (triangle, even hole)-free and has a K_5-minor.

The complexity of the stable set problem and of the coloring problem remain open for even-hole-free graphs.

2. Odd-signable graphs

We *sign* a graph by assigning 0, 1 weights to its edges. A graph is *odd-signable* if there exists a signing that makes the sum of the weights in every chordless cycle (including triangles) odd. Even-hole-free graphs are clearly odd-signable: assign weight 1 to each edge. To characterize odd-signable graphs in terms of excluded induced subgraphs, we now introduce two types of *3-path configurations* (3*PC*'s) and even wheels.

Let x and y be two distinct vertices of G. A $3PC(x, y)$ is a graph induced by three chordless xy-paths, such that any two of them induce a hole. We say that a graph G contains a $3PC(\cdot, \cdot)$ if it contains a $3PC(x, y)$ for some $x, y \in V(G)$. The $3PC(\cdot, \cdot)$'s are also known as *thetas*.

Let $x_1, x_2, x_3, y_1, y_2, y_3$ be six distinct vertices of G such that $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ induce triangles. A $3PC(x_1x_2x_3, y_1y_2y_3)$ is a graph induced by three chordless paths $P_1 = x_1, \ldots, y_1$, $P_2 = x_2, \ldots, y_2$ and $P_3 = x_3, \ldots, y_3$, such that