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Packing chromatic number of cubic graphs

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ABSTRACT

A *packing k-coloring* of a graph G is a partition of $V(G)$ into sets $V_1, \ldots, V_k$ such that for each $1 \le i \le k$ the distance between any two distinct $x, y \in V_i$ is at least $i+1$. The *packing chromatic number*, $\chi_p(G)$, of a graph G is the minimum k such that G has a packing k-coloring. Sloper showed that there are 4-regular graphs with arbitrarily large packing chromatic number. The question whether the packing chromatic number of subcubic graphs is bounded appears in several papers. We answer this question in the negative. Moreover, we show that for every fixed k and $g \ge 2k+2$, almost every n-vertex cubic graph of girth at least g has the packing chromatic number greater than k.

1. Introduction

For a positive integer i, a set S of vertices in a graph G is *i-independent* if the distance in G between any two distinct vertices of S is at least $i+1$. In particular, a 1-independent set is simply an independent set.

A *packing k-coloring* of a graph G is a partition of $V(G)$ into sets $V_1, \ldots, V_k$ such that for each $1 \le i \le k$, the set V_i is i-independent. The *packing chromatic number*, $\chi_p(G)$, of a graph G, is the minimum k such that G has a packing k-coloring. The notion of packing k-coloring was introduced in 2008 by Goddard, Hedetniemi, Hedetniemi, Harris and Rall [15] (under the name *broadcast coloring*) motivated by frequency assignment problems in broadcast networks. The concept has attracted a considerable attention recently: there are more than 25 papers on the topic (see e.g. [1,5–12,14,21] and references in them). In particular, Fiala and Golovach [10] proved that finding the packing chromatic number of a graph is NP-hard even in the class of trees. Sloper [21] showed that there are graphs with maximum degree 4 and arbitrarily large packing chromatic number.

The question whether the packing chromatic number of all *subcubic* graphs (i.e., the graphs with maximum degree at most 3) is bounded by a constant was not resolved. For example, Brešar, Klavžar, Rall, and Wash [7] wrote: *'One of the intriguing problems related to the packing chromatic number is whether it is bounded by a constant in the class of all cubic graphs'*. It was proved in [7,17–19,21] that it is indeed bounded in some subclasses of subcubic graphs. On the other hand, Gastineau and Togni [14] constructed a cubic graph G with $\chi_p(G) = 13$, and asked whether there are cubic graphs with a larger packing chromatic number. Brešar, Klavžar, Rall, and Wash [8] answered this question in affirmative by constructing a cubic graph G' with $\chi_p(G') = 14$. The main result of this paper answers the question in full: Indeed, there are cubic graphs with arbitrarily large packing chromatic number. Moreover, we prove that 'many' cubic graphs have 'high' packing chromatic number:

Theorem 1. *For each fixed integer $k \ge 12$ and $g \ge 2k+2$, almost every n-vertex cubic graph G of girth at least g satisfies $\chi_p(G) > k$.*

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The theorem will be proved in the language of the so-called *Configuration model*, $\mathcal{F}_3(n)$. We will discuss this concept and some important facts on it in the next section. In Section 3 we give upper bounds on the sizes c_i of maximum i-independent sets in almost all cubic n-vertex graphs of large girth. The original plan was to show that for a fixed k and large n, the sum $c_1 + \cdots + c_k$ is less than n. But we were not able to prove it (and maybe this is not true). In Section 4, we give an upper bound on the size of the union of an 1-independent, a 2-independent, and a 4-independent sets which is less than $c_1 + c_2 + c_4$. This allows us to prove Theorem 1 in the last section.

2. Preliminaries

2.1. Notation

We mostly use standard notation. If G is a (multi)graph and $v, u \in V(G)$, then $E_G(v, u)$ denotes the set of all edges in G connecting v and u, $e_G(v, u) := |E_G(v, u)|$, and $\deg_G(v) := \sum_{u \in V(G)\setminus\{v\}} e_G(v, u)$. For $A \subseteq V(G)$, $G[A]$ denotes the sub(multi)graph of G induced by A. The independence number of G is denoted by $\alpha(G)$. For $k \in \mathbb{Z}_{>0}$, $[k]$ denotes the set $\{1, \ldots, k\}$.

2.2. The configuration model

The configuration model is due in different versions to Bender and Canfield [2] and Bollobás [3,4]. Our work is based on the version of Bollobás. Let V be the vertex set of the graph, we are going to associate a 3-element set to each vertex in V. Let n be an even positive integer. Let $V_n = [n]$ and consider the Cartesian product $W_n = V_n \times [3]$. A *configuration/pairing* (of order n and degree 3) is a partition of W_n into $3n/2$ pairs, i.e., a perfect matching of elements in W_n. There are

$$\frac{\binom{3n}{2} \cdot \binom{3n-2}{2} \cdot \ldots \cdot \binom{2}{2}}{(3n/2)!} = (3n-1)!!$$

such matchings. Let $\mathcal{F}_3(n)$ denote the collection of all $(3n-1)!!$ possible pairings on W_n. We project each pairing $F \in \mathcal{F}_3(n)$ to a multigraph $\pi(F)$ on the vertex set V_n by ignoring the second coordinate. Then $\pi(F)$ is a 3-regular multigraph (which may or may not contain loops and multi-edges). Let $\pi(\mathcal{F}_3(n)) = \{\pi(F) : F \in \mathcal{F}_3(n)\}$ be the set of 3-regular multigraphs on V_n. By definition,

each simple graph $G \in \pi(\mathcal{F}_3(n))$ corresponds to $(3!)^n$ distinct pairings in $\mathcal{F}_3(n)$. (1)

We will call the elements of V_n - *vertices*, and of W_n - *points*.

Definition 2. Let $\mathcal{G}_g(n)$ be the set of all cubic graphs with vertex set $V_n = [n]$ and girth at least g and $\mathcal{G}'_g(n) = \{F \in \mathcal{F}_3(n) : \pi(F) \in \mathcal{G}_g(n)\}$.

We will use the following result:

Theorem 3 (*Wormald [22], Bollobás [3]*). *For each fixed $g \geq 3$,*

$$\lim_{n\to\infty} \frac{|\mathcal{G}'_g(n)|}{|\mathcal{F}_3(n)|} = \exp\left\{-\sum_{k=1}^{g-1} \frac{2^{k-1}}{k}\right\}. \quad (2)$$

Remark. When we say that *a pairing F has a multigraph property $\mathcal{A}$*, we mean that $\pi(F)$ has property $\mathcal{A}$.

Since dealing with pairings is simpler than working with labeled simple regular graphs, we need the following well-known consequence of Theorem 3.

Corollary 4 (*[20](Corollary 1.1), [16](Theorem 9.5)*). *For fixed $g \geq 3$, any property that holds for $\pi(F)$ for almost all pairings $F \in \mathcal{F}_3(n)$ also holds for almost all graphs in $\mathcal{G}_g(n)$.*

Proof. Suppose property $\mathcal{A}$ holds for $\pi(F)$ for almost all $F \in \mathcal{F}_3(n)$. Let $\mathcal{H}(n)$ denote the set of graphs in $\mathcal{G}_g(n)$ that do *not* have property $\mathcal{A}$ and $\mathcal{H}'(n) = \{F \in \mathcal{F}_3(n) : \pi(F) \in \mathcal{H}(n)\}$. Let $\mathcal{B}(n)$ denote the set of pairings $F \in \mathcal{F}_3(n)$ such that $\pi(F)$ does not have property $\mathcal{A}$. Then $\mathcal{H}'(n) \subseteq \mathcal{B}(n)$. Hence by the choice of $\mathcal{A}$,

$$\frac{|\mathcal{H}'(n)|}{|\mathcal{F}_3(n)|} \leq \frac{|\mathcal{B}(n)|}{|\mathcal{F}_3(n)|} \to 0 \qquad \text{as } n \to \infty. \quad (3)$$

By (1), we have

$$\frac{|\mathcal{H}(n)|}{|\mathcal{G}_g(n)|} = \frac{|\mathcal{H}(n)|}{|\mathcal{H}'(n)|} \cdot \frac{|\mathcal{H}'(n)|}{|\mathcal{G}'_g(n)|} \cdot \frac{|\mathcal{G}'_g(n)|}{|\mathcal{G}_g(n)|} = \frac{1}{(3!)^n} \cdot \frac{|\mathcal{H}'(n)|}{|\mathcal{G}'_g(n)|} \cdot (3!)^n = \frac{|\mathcal{H}'(n)|}{|\mathcal{G}'_g(n)|}.$$