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Note

A topological lower bound for the chromatic number of a special family of graphs

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ABSTRACT

For studying topological obstructions to graph colorings, Hom-complexes were introduced by Lovász. A graph T is called a test graph if for every graph H, the k-connectedness of $|Hom(T, H)|$ implies $\chi(H) \geq k + 1 + \chi(T)$. The proof of the famous Kneser conjecture is based on the fact that $\mathcal{K}_2$, the complete graph on 2 vertices, is a test graph. This result was extended to all complete graphs by Babson and Kozlov. Their proof is based on generalized nerve lemma and discrete Morse theory.

In this paper, we propose a new topological lower bound for the chromatic number of a special family of graphs. As an application of this bound, we give a new proof of the well-known fact that complete graphs are test graphs.

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1. Introduction

Graph coloring is one of the most challenging and practical topics in combinatorics. A proper (vertex) coloring is an assignment of colors to each vertex of a graph such that no edge connects two identically colored vertices. The smallest number of colors needed for proper coloring of a graph G is the chromatic number, $\chi(G)$. In general, determining the chromatic number of a graph is an arduous problem. Even deciding whether a given planar graph is 3-colorable is NP-complete problem [2]. In other words, it means that no convenient method is known for calculating the chromatic number of an arbitrary graph. This question now arises naturally: Can we at least make a "good" approximation on the number of colors we need? To estimate the chromatic number of a graph, we usually need to go through following steps:

- Giving a proper coloring, to find out how many different colors are **sufficient** for coloring.
- Giving a rigorous argument, to show that how many different colors are **necessary** for coloring.

In other words, in the first part we obtain an upper bound and in the second part, we obtain a lower bound on the chromatic number. Whatever these bounds are closer, our estimation is better!

To provide new (topological) lower bounds on the chromatic number of graphs, Hom complexes were defined by Lovász and they have been extensively studied by many authors, see [4,6,9]. The breakthrough proof of the well-known Kneser conjecture by Lovász [7] implies the following lower bound on the chromatic number of graphs.

Theorem 1. *If $|Hom(\mathcal{K}_2, H)|$ is k-connected, then $\chi(H) \geq k + 3$.*

This result was extended to all complete graphs by Babson and Kozlov, see [6, section 19.2].

Theorem 2. *If $|Hom(\mathcal{K}_r, H)|$ is k-connected, then $\chi(H) \geq k + r + 1$.*

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To provide a better lower bound on the chromatic number of graphs, Lovász made the following conjecture:

Conjecture 3 (*Lovász*). *Let C_{2r+1} be the odd cycle with $2r+1$ vertices. If $|Hom(C_{2r+1}, H)|$ is k-connected, then $\chi(H) \geq k+4$.*

More generally, Björner and Lovász made the following conjecture to generalize the concept of a topological obstruction to graph coloring.

Conjecture 4 (*Björner and Lovász*). *If $|Hom(T, H)|$ is k-connected, then $\chi(H) \geq k + \chi(T) + 1$.*

The first conjecture was confirmed by Babson and Kozlov [1], but the second one was disproved by Hoory and Linial [5]. Regarding Conjecture 4, the following definition was proposed by Kozlov in [6]. A graph T is called a (homotopy) test graph if for every graph H, the k-connectedness of $|Hom(T, H)|$ implies $\chi(H) \geq k+1+\chi(T)$. So in this terminology, complete graphs and odd cycles are test graphs. There are many more test graphs known besides these, see [9,10].

In this paper, we propose a new topological lower bound for the chromatic number of a special family of graphs. Moreover, as an application of this bound, we give a new proof of the fact that complete graphs are test graphs. One can interpret our approach as a combinatorial method to find test graphs. We hope that the proposed technique can be effectively used for finding new test graphs.

The organization of the paper is as follows. In Section 2, we review some standard facts on simplicial complexes, partially ordered sets, and G-spaces. Finally, in Section 3, our main results are stated and proved.

2. Preliminaries and notations

Here and subsequently, G stands for a nontrivial finite group, and its identity element is denoted by e. The following is a brief overview of some of the basic concepts we need.

2.1. G-spaces and G-equivariant maps

If X is a set, then a group action of G on X is a function $G \times X \to X$ denoted $(g, x) \mapsto g \cdot x$, such that $e \cdot x = x$ and $(gh) \cdot x = g \cdot (h \cdot x)$ for all $g, h \in G$ and all $x \in X$. If X is a topological space and G is a topological group, then X is called a G-space if G acts continuously on X. If, moreover, the action is free, i.e, for all $x \in X$, $g \cdot x = x$ implies $g = e$, X is called a free G-space. If X and Y are G-spaces, a continuous map $f : X \to Y$ is a G-equivariant map if $f(g \cdot x) = g \cdot f(x)$ for all $g \in G$ and all $x \in X$.

2.2. Simplicial complexes and G-simplicial complexes

We assume that the reader is familiar with standard definitions and concepts of simplicial complexes. We just recall here the main definitions and notations used in the paper. For more background, see [8,11]. A (finite) simplicial complex $\mathcal{K}$ is a non-empty, hereditary set system of finite sets. That is, $F \in \mathcal{K}$, $F' \subset F$ implies $F' \in \mathcal{K}$ and we have $\emptyset \in \mathcal{K}$. The union of all elements of $\mathcal{K}$ is denoted by $V(\mathcal{K})$. The element of $V(\mathcal{K})$ are called vertices of $\mathcal{K}$, and the elements of $\mathcal{K}$ are called the simplices of $\mathcal{K}$. The dimension of a simplex $\sigma \in \mathcal{K}$ is $\dim(\sigma) = |\sigma| - 1$. The dimension of $\mathcal{K}$ is the maximum dimension of the simplices in $\mathcal{K}$. We denote the geometric realization of $\mathcal{K}$ by $|\mathcal{K}|$. A map $f : V(\mathcal{K}) \to V(\mathcal{L})$ is called simplicial if it maps any simplex to a simplex, that is, $\sigma \in \mathcal{K}$ implies $f(\sigma) \in \mathcal{L}$. Every simplicial mapping $f : \mathcal{K} \to \mathcal{L}$ can be extended linearly to get a continuous mapping $|f| : |\mathcal{K}| \to |\mathcal{L}|$, which is called the affine extension of f.

A simplicial G-complex is a simplicial complex together with an action of G on its vertices that takes simplices to simplices. Note that if $\mathcal{K}$ is a simplicial G-complex then the geometric realization $|\mathcal{K}|$ is a G-space under the natural induced action of G. Moreover, if the induced action of G on $|\mathcal{K}|$ is free, then $\mathcal{K}$ is called a free simplicial G-complex. For two simplicial G-complexes $\mathcal{K}$ and $\mathcal{L}$, a simplicial map $f : V(\mathcal{K}) \to V(\mathcal{L})$ is called a simplicial G-equivariant map if $f(g \cdot x) = g \cdot f(x)$ for all $g \in G$ and all $x \in V(\mathcal{K})$. One can easily see that, the affine extension of a simplicial G-equivariant map is a G-equivariant map.

2.3. Partially ordered sets and G-posets

A partially ordered set or poset is a set and a binary relation $\leq$ such that for all $a, b, c \in P$: $a \leq a$ (reflexivity); $a \leq b$ and $b \leq c$ imply $a \leq c$ (transitivity); and $a \leq b$ and $b \leq a$ imply $a = b$ (anti-symmetry). A pair of elements a, b of a partially order set are called comparable if $a \leq b$ or $b \leq a$. A subset of a poset in which each two elements are comparable is called a chain. A function $f : P \to Q$ between partially ordered sets is order-preserving or monotone, if for all a and b in P, $a \leq_P b$ implies $f(a) \leq_Q f(b)$. The order complex of a poset P is the simplicial complex $\Delta(P)$, whose vertices are the elements of P and whose simplices are all chains in P.

A G-poset is a poset together with a G-action on its elements that preserves the partial order, i.e, $x < y \Rightarrow g \cdot x < g \cdot y$. A G-poset P is called free G-poset, if for all x in X, $g \cdot x = x$ implies $g = e$. One can see that, if P is a free G-poset then its order complex $\Delta(P)$ is a free simplicial G-complex.

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