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Regular Hadamard matrices constructed from Hadamard 2-designs and conference graphs

Dean Crnković

Department of Mathematics, University of Rijeka, Radmile Matejčić 2, 51000 Rijeka, Croatia

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ABSTRACT

Suppose there exists a Hadamard 2-$(m, \frac{m-1}{2}, \frac{m-3}{4})$ design having skew incidence matrix. If there exists a conference graph on $2m-1$ vertices, then there exists a regular Hadamard matrix of order $4m^2$. A conference graph on $2m+3$ vertices yields a regular Hadamard matrix of order $4(m+1)^2$.

1. Introduction

A *Hadamard matrix* of order m is an $(m \times m)$-matrix $H = (h_{i,j})$, $h_{i,j} \in \{-1, 1\}$, satisfying $HH^T = H^TH = mI_m$, where I_m is the identity matrix of order m. The order of a Hadamard matrix must be 1, 2, or a multiple of 4. The Hadamard conjecture proposes that a Hadamard matrix of order $4k$ exists for every positive integer k. A Hadamard matrix is *regular* if the row and column sums are constant. A regular Hadamard matrix is necessarily of order $4k^2$ with constant row sum $2k$. It was conjectured that a regular Hadamard matrix of order $4k^2$ exists for every positive integer k. Constructions of regular Hadamard matrices coming from the theory of Menon–Hadamard difference sets are given in [1,5]. Important results about the existence of regular Hadamard matrices are given by T. Xia, M. Xia and J. Seberry in [12,13]. In [13] they proved the following statement: When $k = 47, 71, 151, 167, 199, 263, 359, 439, 599, 631, 727, 919, 5q_1, 5q_2N, 7q_3$, where q_1, q_2 and q_3 are prime powers such that $q_1 \equiv 1 \ (mod\ 4)$, $q_2 \equiv 5 \ (mod\ 8)$ and $q_3 \equiv 3 \ (mod\ 8)$, $N = 2^a3^bt^2$, a, b=0 or 1, $t \neq 0$ is an arbitrary integer, then there exist regular Hadamard matrices of order $4k^2$.

The existence of some regular Hadamard matrices of order $4p^2$, when p is a prime, $p \equiv 7 \ (mod\ 16)$, is established in [8], and the following theorem is proved in [3].

Theorem 1. *Let p and $2p-1$ be prime powers and $p \equiv 3 \ (mod\ 4)$. Then there exists a regular Hadamard matrix of order $4p^2$.*

The construction employed to prove Theorem 1 uses Paley graphs and Paley designs. In this paper, we generalize this construction using conference graphs and Hadamard designs having skew incidence matrix. In a similar way, we generalize the construction of regular Hadamard matrices given in [4].

2. Hadamard 2-designs

A 2-(v, k, λ) *design* is a finite incidence structure $(\mathcal{P}, \mathcal{B}, I)$, where $\mathcal{P}$ and $\mathcal{B}$ are disjoint sets and $I \subseteq \mathcal{P} \times \mathcal{B}$, with the following properties:

E-mail address: deanc@math.uniri.hr.

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1. $|\mathcal{P}| = v$;
2. every element of $\mathcal{B}$ is incident with exactly k elements of $\mathcal{P}$;
3. every pair of distinct elements of $\mathcal{P}$ is incident with exactly λ elements of $\mathcal{B}$.

The elements of the set $\mathcal{P}$ are called *points* and the elements of the set $\mathcal{B}$ are called *blocks*. If $|\mathcal{P}| = |\mathcal{B}| = v$ and $2 \leq k \leq v-2$, then a 2-(v, k, λ) design is called a *symmetric design*.

Hadamard matrices of order $4k$ can be used to create symmetric designs with parameters $(4k-1, 2k-1, k-1)$ or $(4k-1, 2k, k)$, which are called *Hadamard 2-designs* (see [2,11]). The construction is reversible, so that symmetric designs with these parameters can be used to construct Hadamard matrices. It is also well known that the existence of a symmetric design with parameters $(4u^2, 2u^2-u, u^2-u)$ or $(4u^2, 2u^2+u, u^2+u)$ is equivalent to the existence of a regular Hadamard matrix of order $4u^2$ (see [11, Theorem 1.4]). Such symmetric designs are called *Menon designs*.

A matrix A is *skew-symmetric* if $A^T = -A$. A Hadamard matrix H of order $4k$ is *skew-type* if $H = A + I_{4k}$, where $A^T = -A$. A $(0, 1)$-matrix D is *skew* if $D + D^T = J - I$, where J the all-one matrix. A skew-type Hadamard matrix corresponds to a Hadamard 2-design with skew incidence matrix, and vice versa.

Suppose D is the incidence matrix of a symmetric design. $D + I$ is again the incidence matrix of a symmetric design if and only if D is skew (see [9]). Let D be a skew incidence matrix of a Hadamard 2-$(m, \frac{m-1}{2}, \frac{m-3}{4})$ design. Further, let $\overline{D} = (\overline{d}_{ij})$ be an incidence matrix of a complementary Hadamard design with parameters $(m, \frac{m+1}{2}, \frac{m+1}{4})$. Then $D + I_m$ and $\overline{D} - I_m$ are incidence matrices of symmetric designs with parameters $(m, \frac{m+1}{2}, \frac{m+1}{4})$ and $(m, \frac{m-1}{2}, \frac{m-3}{4})$, respectively.

For $v \in N$ we denote by j_v the all-one vector of dimension v, by 0_v the zero-vector of dimension v, by J_v the all-one matrix of dimension $(v \times v)$, and by $0_{v\times v}$ the zero-matrix of dimension $(v \times v)$. Further, for $a, b \in N$ we denote by $J_{a\times b}$ the all-one matrix of dimension $(a \times b)$.

Lemma 1. *Let D be a skew incidence matrix of a Hadamard* 2-$(m, \frac{m-1}{2}, \frac{m-3}{4})$ *design and let $\overline{D} = (\overline{d}_{ij})$ be an incidence matrix of a complementary Hadamard $(m, \frac{m+1}{2}, \frac{m+1}{4})$ design. The matrices D and $\overline{D}$ have the following properties:*

$$D \cdot \overline{D}^T = (\overline{D} - I_m)(D + I_m)^T = \frac{m+1}{4} J_m - \frac{m+1}{4} I_m,$$

$$[\, D \mid \overline{D} - I_m \,] \cdot [\, \overline{D} - I_m \mid D \,]^T = \frac{m-1}{2} J_m - \frac{m-1}{2} I_m,$$

$$[\, D \mid D \,] \cdot [\, D + I_m \mid \overline{D} - I_m \,]^T = \frac{m-1}{2} J_m,$$

$$[\, \overline{D} \mid D \,] \cdot [\, \overline{D} - I_m \mid \overline{D} - I_m \,]^T = \frac{m-1}{2} J_m.$$

Proof. Each block of a symmetric $(m, \frac{m-1}{2}, \frac{m-3}{4})$ design meets the complement of any other block in $\frac{m+1}{4}$ points. Hence, $D \cdot \overline{D}^T = \frac{m+1}{4} J_m - \frac{m+1}{4} I_m$. Further, $D + I_m$ is the incidence matrix of a symmetric $(m, \frac{m+1}{2}, \frac{m+1}{4})$ design and $\overline{D + I_m} = \overline{D} - I_m$, so $(\overline{D} - I_m)(D + I_m)^T = \frac{m+1}{4} J_m - \frac{m+1}{4} I_m$.

The other equalities follow from the properties listed below:

$$D_i \cdot (\overline{D} - I_m)_j^T = \begin{cases} 0, & \text{if } i = j, \\ \dfrac{m+1}{4} - 1, & \text{if } d_{ij} = 1, \\ \dfrac{m+1}{4}, & \text{if } d_{ij} = 0, i \neq j, \end{cases}$$

$$(\overline{D} - I_m)_i \cdot D_j^T = \begin{cases} 0, & \text{if } i = j, \\ \dfrac{m+1}{4} - 1, & \text{if } d_{ji} = 1, \\ \dfrac{m+1}{4}, & \text{if } d_{ji} = 0, i \neq j, \end{cases}$$

$$D_i \cdot (D + I_m)_j^T = \begin{cases} \dfrac{m-1}{2}, & \text{if } i = j, \\ \dfrac{m-3}{4} + 1, & \text{if } d_{ij} = 1, \\ \dfrac{m-3}{4}, & \text{if } d_{ij} = 0, i \neq j, \end{cases}$$

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