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Existence of incomplete canonical Kirkman packing designs

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ABSTRACT

For u, v positive integers with $u \equiv v \equiv 4 \pmod 6$, let ICKPD$(u, v)$ denote a canonical Kirkman packing of order u missing one of order v. In this paper, it is shown that the necessary condition for existence of an ICKPD(u, v), namely $u \geq 3v + 4$, is sufficient with a definite exception $(u, v) = (16, 4)$, and except possibly when $v > 76$, $v \equiv 4 \pmod{12}$ and $u \in \{3v + 4, 3v + 10\}$.

1. Introduction

A packing of order v is a pair $(X, \mathscr{B})$ where X is a v-set and $\mathscr{B}$ is a collection of subsets (called blocks) of X such that each 2-subset of X is contained in at most one block of $\mathscr{B}$. The leave of $(X, \mathscr{B})$ is a graph (X, E) where $\{x, y\} \in E$ if and only if $\{x, y\}$ is not contained in any block of $\mathscr{B}$. A packing is called resolvable if its block set $\mathscr{B}$ admits a partition into parallel classes, each parallel class being a partition of the v-set X.

When $v \equiv 3 \pmod 6$, the maximum possible number of parallel classes in a resolvable packing of a v-set by triples cannot exceed $(v-1)/2$. A resolvable packing of a v-set by triples that achieves this bound is called a Kirkman triple system, and is denoted by KTS(v). The leave of a KTS(v) is an empty set.

Similarly, if $v \equiv 0 \pmod 6$, then a resolvable packing of a v-set by triples with $v/2 - 1$ parallel classes is called a nearly Kirkman triple system, denoted by NKTS(v). The leave of an NKTS(v) is a 1-factor.

For the existence of Kirkman triple systems and nearly Kirkman triple systems, we have the following results:

Theorem 1.1 ([20]). *There exists a KTS(v) if and only if $v \equiv 3 \pmod 6$.*

Theorem 1.2 ([3,4,18,21]). *There exists an NKTS(v) if and only if $v \equiv 0 \pmod 6$ and $v \geq 18$.*

For a given $v \equiv 4 \pmod 6$ and $v \geq 4$, following [8], we define a canonical Kirkman packing design of order v, denoted by CKPD(v), to be a resolvable packing with $(v-4)/2$ parallel classes such that:

(i) each parallel class consists of a size 4 block and $(v-4)/3$ triples;

(ii) the leave consists of the union of $(v-4)/2$ vertex-disjoint edges and a K_4 with no vertices in common with those edges.

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It is clear that a CKPD(v) is equivalent to a resolvable {3, 4}-GDD of type $2^{(v-4)/2}4^1$ in which each parallel class contains exactly one block of size 4 [19, Proposition 2.1]. It is also clear that no point in the size 4 group can occur in any block of size 4, and every other point appears in two size 4 blocks.

For information on some types of Kirkman packing designs that are not canonical Kirkman packings, the reader is referred to [5–7]. The existence of canonical Kirkman packing designs has been completely determined:

Theorem 1.3 ([8,10,19]). *Let $v \equiv 4 \pmod 6$. Then there exists a CKPD(v) if and only if $v \geq 22$.*

For given positive integers u and v with $u, v \equiv 4 \pmod 6$, let $(X, \mathscr{A})$ be a CKPD(u) and $(Y, \mathscr{B})$ be a CKPD(v). If $Y \subseteq X$, $\mathscr{B} \subseteq \mathscr{A}$, each parallel class of $(Y, \mathscr{B})$ is a part of some parallel class of $\mathscr{A}$, and the leave of $(Y, \mathscr{B})$ is a subgraph of the leave of $(X, \mathscr{A})$, then we say $(Y, \mathscr{B})$ is embedded in $(X, \mathscr{A})$, or $(Y, \mathscr{B})$ is a subsystem of $(X, \mathscr{A})$. Removing all the blocks of $\mathscr{B}$ from $\mathscr{A}$ gives an incomplete canonical Kirkman packing design. Formally, we give the following definition:

Let $u, v \equiv 4 \pmod 6$. An incomplete canonical Kirkman packing design of order u with a hole of size v, denoted by ICKPD(u, v), is a triple $(X, Y, \mathscr{C})$ where X is a point set of u elements, Y (called a hole) is a v-subset of X, and $\mathscr{C}$ is a collection of subsets (blocks) of X such that:

(i) $|B \cap Y| \leq 1$ for each $B \in \mathscr{C}$;

(ii) any two distinct elements of X occur together in at most one block;

(iii) $\mathscr{C}$ admits a partition into $(u-v)/2$ parallel classes on X, each of which contains one block of size 4 and $(u-4)/3$ triples, and further $(v-4)/2$ holey parallel classes of triples, each of which contains each element of $X \setminus Y$ once and no element of the hole Y.

(iv) each element of $X \setminus Y$ is contained in exactly two blocks of size 4.

The leave of an ICKPD(u, v) consists of the union of a K_v on Y and $(u-v)/2$ vertex-disjoint edges on $X \setminus Y$.

The embedding problem for various kinds of resolvable designs has been studied extensively and completely solved for Kirkman triple systems and nearly Kirkman triple systems.

Theorem 1.4 ([22,24]). *A KTS(v) can be embedded in a KTS(u) if and only if $u, v \equiv 3 \pmod 6$ and $u \geq 3v$.*

Theorem 1.5 ([11–13]). *An NKTS(v) can be embedded in an NKTS(u) if and only if $u, v \equiv 0 \pmod 6$, $v \geq 18$ and $u \geq 3v$. Further, if $v \in \{6, 12\}$ and $u \geq 3v$, then there exists an NKTS(u) with a hole of size v.*

In 2008, Deng and Su [14] investigated the problem of embedding of CKPDs and established the following results.

Lemma 1.6 ([14]). *Let $u, v \equiv 4 \pmod 6$. If a CKPD(v) can be embedded in a CKPD(u), then $u \geq 3v + 4$.*

Lemma 1.7 ([14]). *Let $u, v \equiv 4 \pmod 6$, $v \geq 82$. Then there exists an ICKPD(u, v) for $u \geq 3.5v$.*

Lemma 1.8 ([14]). *Let $v \in \{4, 10\}$. Then there exists an ICKPD(u, v) whenever $u \equiv 4 \pmod 6$, $u \geq 3v + 4$ with the definite exception $(u, v) = (16, 4)$.*

Recently, Cheng and Wang [9] have investigated the existence of ICKPD(u, v)s further and obtained:

Lemma 1.9 ([9]). *Let $v = 16$ or 22. Then there exists an ICKPD(u, v) if and only if $u \equiv 4 \pmod 6$ and $u \geq 3v + 4$ except possibly for $(u, v) = (52, 16)$.*

In this paper, we will give more constructions of ICKPD(u, v)s.

The remainder of this paper is organized as follows. Section 2 gives the basic concepts and construction methods for GDDs and constructs directly some new non-uniform 4-GDDs which will be used later on. Section 3 shows existence of ICKPD(u, v)s for small v ($4 \leq v \leq 76$). Section 4 deals with the existence problem for ICKPD(u, v)s with maximum holes. Section 5 is devoted to determining the spectrum for ICKPDs. A brief conclusion will be given in Section 6.

2. Some new 4-GDDs

In this section, we will use direct and recursive constructions to give some new non-uniform 4-GDDs for later use.

A group divisible design (GDD) is a triple $(X, \mathscr{G}, \mathscr{B})$ where X is a set of points, $\mathscr{G}$ is a partition of X into groups, and $\mathscr{B}$ is a collection of subsets (blocks) of X so that any pair of distinct points occurs either in some group or in exactly one block, but not both. A K-GDD $(X, \mathscr{G}, \mathscr{B})$ of type $g_1^{u_1} g_2^{u_2} \cdots g_s^{u_s}$ is a GDD with u_i groups of size g_i, $i = 1, 2, \ldots, s$ and whose block sizes all lie in the set K. A GDD is called *uniform* if all of its groups have the same size. Also, a K-GDD with $K = \{k\}$ is usually called a k-GDD.

The spectra for uniform 4-GDDs and 4-GDDs of type $g^4 m^1$ have been determined, see [16, IV.4.1, Theorems 4.6 and 4.12].