ARTICLE IN PRESS
Discrete Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

List neighbor sum distinguishing edge coloring of subcubic graphs

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ARTICLE INFO

Article history:
Received 22 April 2017
Received in revised form 19 September 2017
Accepted 20 September 2017
Available online xxxx

Keywords:
List neighbor sum distinguishing edge coloring
Combinatorial Nullstellensatz
Subcubic graph

ABSTRACT

A proper k-edge-coloring of a graph with colors in $\{1, 2, \ldots, k\}$ is neighbor sum distinguishing (or, NSD for short) if for any two adjacent vertices, the sums of the colors of the edges incident with each of them are distinct. Flandrin et al. conjectured that every connected graph with at least 6 vertices has an NSD edge coloring with at most $\Delta+2$ colors. Huo et al. proved that every subcubic graph without isolated edges has an NSD 6-edge-coloring. In this paper, we first prove a structural result about subcubic graphs by applying the decomposition theorem of Trotignon and Vušković, and then applying this structural result and the Combinatorial Nullstellensatz, we extend the NSD 6-edge-coloring result to its list version and show that every subcubic graph without isolated edges has a list NSD 6-edge-coloring.

1. Introduction

All graphs considered in this paper are finite and simple. We follow the standard notation and terminology in [4]. Let $k \geq 1$ be an integer and $[k] = \{1, 2, \ldots, k\}$. A proper *$k$-edge-coloring* of a graph G is a mapping $\phi : E(G) \to [k]$ such that any two adjacent edges receive different colors. The coloring ϕ is called *neighbor sum distinguishing* (or, NSD for short) if for each $uv \in E(G)$, $\sum_{e\in\partial_G(u)}\phi(e) \neq \sum_{e\in\partial_G(v)}\phi(e)$, where $\partial_G(v)$ is the set of edges incident with v in G. The *NSD chromatic number* of G is the smallest integer k such that G has an NSD k-edge-coloring, denoted by $\chi'_{\Sigma}(G)$. Note that a graph has an NSD chromatic number if and only if it does not contain isolated edges. Such graph is said to be *normal*. Flandrin et al. [6] first introduced the NSD k-edge-coloring of graphs and proposed the following conjecture.

Conjecture 1.1 (*Flandrin et al. [6]*). *$\chi'_{\Sigma}(G) \leq \Delta(G) + 2$ for a connected graph G with $|V(G)| \geq 6$.*

In the same paper, Flandrin et al. confirmed Conjecture 1.1 for cycles, trees, hypercubes, complete graphs and complete bipartite graphs. They also proved $\chi'_{\Sigma}(G) \leq \lceil\frac{7\Delta(G)-4}{2}\rceil$ for any graph G with $\Delta(G) \geq 2$ and $\chi'_{\Sigma}(G) \leq 8$ if G is cubic. Wang and Yan [20] reduced the upper bound to $\lceil\frac{10\Delta(G)+2}{3}\rceil$ when $\Delta(G) \geq 18$. By applying the probabilistic method, Przybyło [15] reduced the upper bound to $(1 + o(1))\Delta(G)$ when $\Delta(G)$ is sufficiently large. Recently, Huo et al. [10] further reduced the upper bound for subcubic graphs by proving that every normal subcubic graph has $\chi'_{\Sigma}(G) \leq 6$. The readers are referred to [3,5,7,8,19] for more results on NSD edge coloring.

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https://doi.org/10.1016/j.disc.2017.09.023

The neighbor sum distinguishing edge coloring is also inspired and connected to a similar concept without the requirement of proper coloring (called NSD k-edge-weighting) introduced by Karoński, Łuczak and Thomason [12] who proposed the following famous 1-2-3 conjecture.

Conjecture 1.2 (*Karoński, Łuczak and Thomason [12], 1-2-3 Conjecture*)**.** *Every connected graph with at least three vertices has an NSD* 3*-edge-weighting.*

Kalkowski, Karoński, and Pfender [11] showed that every connected graph with at least three vertices has an NSD 5-edge-weighting. Karoński, Łuczak, and Thomason [12] also showed that 1-2-3 Conjecture is true for 3-colorable graphs. Readers are referred to the survey paper [17] for more information.

Both NSD edge coloring and NSD edge weighting problems have been studied for their list versions. Przybyło [14] first studied the list version of NSD edge coloring. Bartnicki et al. [2] first studied the edge weighting choosability and conjectured that the 1-2-3 Conjecture is true for its list version.

A *k-list assignment* of a graph G is a mapping L that assigns a list $L(e)$ of k real numbers to each edge e. An *L-edge-coloring* of G is a mapping $\phi : E(G) \to \cup_{e\in E(G)} L(e)$ such that the color of each edge e is in $L(e)$ and different from the colors of all edges adjacent to e. Clearly, the L-edge-coloring is a generalization of k-edge-coloring. Similarly, the *NSD choice number* $\mathrm{ch}'_{\Sigma}(G)$ of G is defined as the smallest integer k such that G has an NSD L-edge-coloring for any k-list assignment L.

Przybyło [14] proved that $\mathrm{ch}'_{\Sigma}(G) \le 2\Delta(G) + \mathrm{col}(G) - 1$, where $\mathrm{col}(G)$ is the least integer k such that G has a vertex ordering in which each vertex is preceded by fewer than k of its neighbors. Later, Przybyło and Wong [16] revised this upper bound and proved the following theorem.

Theorem 1.3 (*Przybyło and Wong [16]*)**.** *Let G be a graph without isolated edges. Then* $\mathrm{ch}'_{\Sigma}(G) \le \Delta(G) + 3\mathrm{col}(G) - 4$.

In [9], Huo et al. investigated a relaxed case of NSD L-edge-coloring by requiring that each list $L(e)$ consists of k positive real numbers, and used $\mathrm{ch}'_{\Sigma^p}(G)$ to denote the parameter corresponding to $\mathrm{ch}'_{\Sigma}(G)$. Let $\mathrm{mad}(G)$ denote the maximum average degree of G. They proved that every normal subcubic graph G has $\mathrm{ch}'_{\Sigma^p}(G) \le 7$, and $\mathrm{ch}'_{\Sigma^p}(G) \le 6$ if $\mathrm{mad}(G) < \frac{36}{13}$. It is easy to see that $\chi'_{\Sigma}(G) \le \mathrm{ch}'_{\Sigma^p}(G) \le \mathrm{ch}'_{\Sigma}(G)$. Very recently, Yu et al. [21] proved that $\mathrm{ch}'_{\Sigma}(G) \le 6$ if G is a normal subcubic graph with at least two 2-vertices.

In this paper, we first show that every connected subcubic graph must contain some special induced subgraphs by applying the decomposition theorem of Trotignon and Vušković [18] and then applying this structural result and the Combinatorial Nullstellensatz, we improve the results in [9,10] and [21] and prove the following result.

Theorem 1.4. *Every normal subcubic graph G has* $\mathrm{ch}'_{\Sigma}(G) \le 6$.

Before proceeding, we introduce some notations and terminology. Let $G = (V(G), E(G))$ be a graph. For two disjoint vertex subsets V_1 and V_2, denote by $E_G(V_1, V_2)$ the set of edges of G with one end in V_1 and the other end in V_2. In particular, if $V_2 = V(G) \setminus V_1$, we write $E_G(V_1, V_2)$ as $\partial_G(V_1)$, and if $V_1 = \{v\}$, we write $\partial_G(\{v\})$ as $\partial_G(v)$. We refer to $|\partial_G(v)|$ as the *degree* of v in G and denote it by $d_G(v)$. If $d_G(v) = \ell$ (resp., $d_G(v) \le \ell$), we call v an *ℓ-vertex* (resp., *ℓ^--vertex*). G is *subcubic* if every vertex is a 3^--vertex. The maximum degree of G is denoted by $\Delta(G)$.

2. A structural lemma for subcubic graphs

In this section, we will show that every connected subcubic graph must contain some special induced subgraphs by applying the decomposition theorem of Trotignon and Vušković [18].

Before we state the decomposition theorem of Trotignon and Vušković, we need to introduce the following definitions. Let G be a graph.

- G is *strongly* 2*-bipartite* if it is square-free and bipartite with bipartition (X, Y) where X is the set of all 2-vertices of G and Y is the set of all ℓ-vertices of G with $\ell \ge 3$.
- A *k-cut* of G is a subset $S \subseteq V(G)$ with $|S| = k$ such that $V(G) \setminus S$ can be partitioned into non-empty sets X, Y with $E_G(X, Y) = \emptyset$. A k-cut S is *proper* if S is a stable set consisting of vertices of degree at least 3, $|X| \ge 2$, $|Y| \ge 2$, and both $G[X \cup S]$ and $G[Y \cup S]$ are connected. (X, Y, S) is called a split of this proper k-cut.
- A *proper* 1*-join* of G is a partition of $V(G)$ into sets X, Y such that there exist sets $A \subseteq X$ and $B \subseteq Y$ satisfying:
  ◦ $|A| \ge 2$ and $|B| \ge 2$;
  ◦ $G[A \cup B]$ is a complete bipartite graph with partitions A and B;
  ◦ $E_G(X, Y) = E_G(A, B)$.
  (X, Y, A, B) is called a split of this proper 1-join.

Theorem 2.1 (*Trotignon and Vušković [18]*)**.** *Let G be a connected graph containing no cycle with a unique chord as an induced subgraph. Then either G is a strongly* 2*-bipartite, or G is a cycle of length at least* 7*, or G is a clique, or G is an induced subgraph of the Petersen graph P_{10} (Fig. 1(a)) or the Heawood graph H_{14} (Fig. 1(b)), or G has a* 1*-cut, a proper* 2*-cut, or a proper* 1*-join.*

Please cite this article in press as: Y. Lu, et al., List neighbor sum distinguishing edge coloring of subcubic graphs, Discrete Mathematics (2017), https://doi.org/10.1016/j.disc.2017.09.023.