Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

# On nice and injective-nice tournaments

T.K. Rapke [a], K. Seyffarth [b,*]

[a] *Faculty of Education, York University, Toronto, ON M3J 1P3, Canada*
[b] *Department of Mathematics and Statistics, University of Calgary, Calgary, AB T2N 1N4, Canada*

ARTICLE INFO

*Article history:*
Received 7 July 2016
Received in revised form 6 June 2017
Accepted 8 June 2017
Available online xxxx

*Keywords:*
Homomorphisms
Oriented colouring
Injective oriented colouring
Nice digraphs
Tournaments

ABSTRACT

One way of defining an oriented colouring of a directed graph $\vec{G}$ is as a homomorphism from $\vec{G}$ to a target directed graph $\vec{H}$, and an injective oriented colouring of $\vec{G}$ can be defined as a homomorphism from $\vec{G}$ to a target directed graph $\vec{H}$ such that no two in-neighbours of a vertex of $\vec{G}$ have the same image. Oriented colourings may be constructed using target directed graphs that are *nice*, as defined by Hell et al. (2001). We extend the work of Hell et al. by considering target graphs that are tournaments, characterizing nice tournaments, and proving that every nice tournament on $n$ vertices is $k$-nice for some $k \leq n+2$. We also give a characterization of tournaments that are nice but not injective-nice.


## 1. Introduction

An oriented colouring of a directed graph $\vec{G}$ is a colouring of the vertices of $\vec{G}$ so that the vertices of each arc receive different colours, and furthermore, all arcs between sets of vertices of different colours, say red and green, have the same direction (for example, are directed from red vertices to green vertices). Such a colouring is injective if, in addition, no two arcs directed into a vertex originate at vertices with the same colour as each other. Equivalent definitions of both oriented colourings and injective oriented colourings can be given in terms of homomorphisms of directed graphs. The reader may wish to consult *Graphs and Homomorphisms* [5] as a great reference on homomorphisms. A *homomorphism* from a directed graph $\vec{G}$ to a directed graph $\vec{H}$ is a mapping from the vertex set of $\vec{G}$ to the vertex set of $\vec{H}$ that preserves arcs. An *injective homomorphism* is a homomorphism in which no two in-neighbours of a vertex of $\vec{G}$ have the same image. Here, we consider directed graphs that are oriented graphs (i.e., directed graphs obtained from simple loopless graphs by assigning one of two possible directions to each edge). An *oriented colouring* of an oriented graph $\vec{G}$ is a homomorphism from $\vec{G}$ to a target oriented graph $\vec{H}$, and an *injective oriented colouring* of an oriented graph $\vec{G}$ is an injective homomorphism from $\vec{G}$ to a target oriented graph $\vec{H}$. The *oriented chromatic number* of an oriented graph $\vec{G}$ is the minimum number of vertices in a target oriented graph $\vec{H}$ so that there exists a homomorphism from $\vec{G}$ to $\vec{H}$. The *injective oriented chromatic number* of an oriented graph $\vec{G}$ is the minimum number of vertices in a target oriented graph $\vec{H}$ so that there exists an injective homomorphism from $\vec{G}$ to $\vec{H}$.

Obtaining upper bounds on oriented chromatic numbers of oriented graphs has proven to be a challenging problem that has attracted many authors [2,3,9–13]. For the oriented graphs considered by these authors, oriented colourings are often constructed by using target oriented graphs having the property that Hell, Kostochka, Raspaud and Sopena [4] have termed *nice*. Since an oriented graph $\vec{H}$ can be embedded in a tournament $\vec{H}'$, we focus our attention on nice tournaments as potential targets. The properties of tournaments allow us to determine the specific structure of target tournaments that are not nice. We prove (Theorem 12) that a tournament is not nice if and only if it is either not strongly connected, or is strongly connected

* Corresponding author.
*E-mail addresses:* trapke@edu.yorku.ca (T.K. Rapke), kseyffar@ucalgary.ca (K. Seyffarth).

http://dx.doi.org/10.1016/j.disc.2017.06.006
0012-365X/

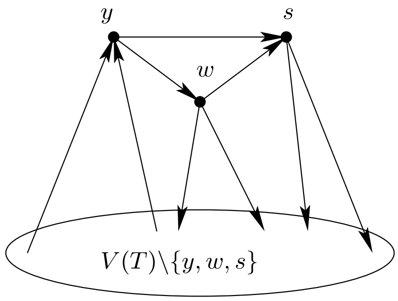


**Fig. 1.** Structure of nice tournaments that are not injective-nice.

and contains two vertices $x$ and $y$ such that $x$ has $y$ as its only out-neighbour and $y$ has $x$ as its only in-neighbour. Theorem 12 implies that there are no nice tournaments on four or fewer vertices. Using the characterization in Theorem 12, we prove that every nice tournament on $n$ vertices is $k$-nice for some $k \leq n+2$.

Recently, there has been interest in injective oriented colourings [6,7,12]. We consider target tournaments that have the property that we term *injective-nice*, and can be considered to be the "injective" analogue of nice directed graphs. We prove (Theorem 36) that a nice tournament $T$ is not injective-nice if and only if it contains three vertices $y$, $w$ and $s$ such that $y$ has $w$ and $s$ as its only out-neighbours, $s$ has $y$ and $w$ as its only in-neighbours, and $w$ has $y$ as its only in-neighbour (see Fig. 1). This could have applications to obtaining bounds on injective oriented chromatic numbers of planar graphs with sufficiently large girth, or graphs with sufficiently small maximum average degree. Bounds on the oriented chromatic numbers of graphs in these two classes were obtained using the fact that these graphs have the property that Hell et al. [4] subsequently termed $k$-nice. Similar bounds on injective oriented chromatic numbers may be achievable for graphs in these classes (and possibly others) using $k$-injective-nice tournaments.

## 2. Notation and preliminary results

We use the terminology and notation of Bondy and Murty [1] unless otherwise specified. A *tournament* $T$ is a complete simple graph (no loops or multiple edges) with one of two possible directions assigned to each edge. Then $T$ is a directed graph, or simply *digraph*, consisting of a set of vertices $V(T)$, and a set of ordered pairs of vertices called arcs, $A(T)$, where $\overrightarrow{xy}$ denotes an arc from $x$ to $y$. For $X \subseteq V(T)$, we let $T[X]$ denote the subgraph of $T$ induced by $X$. Let $\vec{G}$ be a digraph with underlying simple graph $G$. A *walk* in $\vec{G}$ is a sequence of vertices $x_0x_1\cdots x_n$ such that $x_{i-1}x_i \in E(G)$ for each $i$, $1 \leq i \leq n$. A *directed path* in $\vec{G}$ is a sequence $x_0x_1\cdots x_n$ of distinct vertices of $V(\vec{G})$ such $\overrightarrow{x_{i-1}x_i} \in A(\vec{G})$ for each $i$, $1 \leq i \leq n$. A digraph $\vec{G}$ is *strongly connected* if there is a directed path between any pair of distinct vertices.

Terminology similar to that of Hell et al. [4] is used when discussing nice tournaments and their properties. Let $Q = q_1q_2\ldots q_k$ be a pattern of length $k$ over $\{+,-\}$. A $Q$-*walk* in a digraph $\vec{G}$ is defined to be a walk $P = x_0x_1\ldots x_{k-1}x_k$ such that for each $i$, $1 \leq i \leq k$, $\overrightarrow{x_{i-1}x_i} \in A(\vec{G})$ if $q_i = +$ and $\overrightarrow{x_ix_{i-1}} \in A(\vec{G})$ if $q_i = -$. For $X \subseteq V(\vec{G})$, we let $N_Q(X)$ denote the set of all vertices $y$ such that there exists a $Q$-walk from some $x \in X$ to $y$. A digraph $\vec{G}$ is $k$-*nice* if for every pattern $Q$ of length $k$ and every vertex $x \in V(\vec{G})$ we have $N_Q(x) = V(\vec{G})$. Note that if $\vec{G}$ is $k$-nice, then $\vec{G}$ is also $(k+1)$-nice. A digraph is *nice* if it is $k$-nice for some $k$. Hell et al. [4] characterize nice digraphs in terms of a structure they define as a *black hole*. Suppose $X \subset V(\vec{G})$, $X \neq \emptyset$, and suppose $Q$ is a non-empty pattern over $\{+,-\}$. If $N_Q(X) \subseteq X$ then the pair $(X, Q)$ is a called a *black hole*. In what follows, we rely heavily on the following consequence of Hell et al. [4, Proposition 2].

**Proposition 1.** *A digraph $\vec{G}$ is not nice if and only if $\vec{G}$ has a black hole.*

In the context of injective homomorphisms, we define an analogue of nice, namely *injective-nice*, and adapt the definitions provided by Hell et al. [4]. We define an *injective $Q$-walk* of length $k$ in digraph $\vec{G}$ to be a $Q$-walk of length $k$, $x_0x_1\cdots x_k$, with the additional property that for all substrings $q_jq_{j+1} = +-$, $1 \leq j \leq k-1$, $x_{j-1} \neq x_{j+1}$. For $X \subseteq V(\vec{G})$, we let $\widehat{N}_Q(X)$ denote the set of all vertices $y$ such that there exists an injective $Q$-walk from some $x \in X$ to $y$. Note that there is no difference between a $Q$-walk and an injective $Q$-walk when the pattern $Q$ does not contain the substring $+-$. For this reason, we often write $N_Q(X)$ instead of $\widehat{N}_Q(X)$ when $Q$ does not contain $+-$.

For a digraph $\vec{G}$ and $X \subset V(\vec{G})$, $N_{+-}(X) = N_-(N_+(X))$ whereas it may not be the case that $\widehat{N}_{+-}(X) = N_-(N_+(X))$; i.e., we could have $N_{+-}(X) \neq \widehat{N}_{+-}(X)$. For example, in the tournament $\mathcal{F}_4$ shown in Fig. 2, $N_{+-}(y) = N_-(\{w, s\}) = \{y, w\}$, while $\widehat{N}_{+-}(y) = \{w\}$. In this context, we say that $y$ is a *lost vertex*; more generally, $N_{+-}(X) \setminus \widehat{N}_{+-}(X)$ is a set of *lost vertices* (with respect to $X$). The existence of lost vertices implies that it is possible to have $N_+(X) = V(\vec{G})$ while $\widehat{N}_{+-}(X) \neq V(\vec{G})$. Again, in