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Domination in the hierarchical product and Vizing's conjecture

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ABSTRACT

Given a graph G, a set $S \subseteq V(G)$ is a dominating set of G if every vertex of G is either in S or adjacent to a vertex in S. The domination number of G, denoted $\gamma(G)$, is the minimum cardinality of a dominating set of G. Vizing's conjecture states that $\gamma(G\Box H) \geq \gamma(G)\gamma(H)$ for any graphs G and H where $G\Box H$ denotes the Cartesian product of G and H. In this paper, we continue the work by Anderson et al. (2016) by studying the domination number of the hierarchical product. Specifically, we show that partitioning the vertex set of a graph in a particular way shows a trend in the lower bound of the domination number of the product, providing further evidence that the conjecture is true.

1. Introduction

Over the past five decades, a common approach to Vizing's conjecture is to show that if G is a specific type of graph, then $\gamma(G\Box H) \geq \gamma(G)\gamma(H)$ for any graph H, in which case we say that G satisfies "Vizing's inequality". Jacobson and Kinch [10] proved in 1986 that trees satisfy Vizing's inequality, and El-Zahar and Pareek [8] showed in 1991 that cycles do as well. Sun in 2004 [12], and later Brešar in 2013 [5] with a different proof, showed the inequality holds for any graph with domination number 3. The best lower bound to date for $\gamma(G\Box H)$ was shown to be $\frac{1}{2}\gamma(G)\gamma(H)$ by Clark and Suen [7]. This lower bound was later improved to $\frac{1}{2}\gamma(G)\gamma(H) + \frac{1}{2}\min\{\gamma(G), \gamma(H)\}$ by Suen and Tarr [11].

One of the more successful approaches to this problem involves partitioning the vertex set of a graph G in order to "keep track" of dominating sets in the product. As early as 1979, Barcalkin and German [2] showed that if $V(G)$ can be partitioned into $\gamma(G)$ sets each of which induces a clique, then G satisfies Vizing's inequality. In a similar vein, Hartnell and Rall [9] in 1995 identified the class of Type χ graphs which is arguably the largest class of graphs to date for which the inequality holds. In this paper, we show that if G has domination 4 and $V(G)$ can be partitioned in a particular way, then $\gamma(G\Box H) \geq 3\gamma(H)$ for any graph H.

Another more recent approach to the conjecture was given in [1] where domination in the generalized hierarchical product of two graphs G and H, denoted $G(U) \sqcap H$, was explored. This relatively new graph product is a generalization of the Cartesian product. In this paper, we provide a lower bound for $\gamma(G(U) \sqcap H)$ and we argue that the bound provides further evidence that Vizing's conjecture is true.

The remainder of this paper is organized as follows. In Section 1.1, we provide definitions and terminology that will be used throughout the paper. In Section 2, we study domination in the generalized hierarchical product. Finally, we give yet another approach to attacking the conjecture in Section 3.

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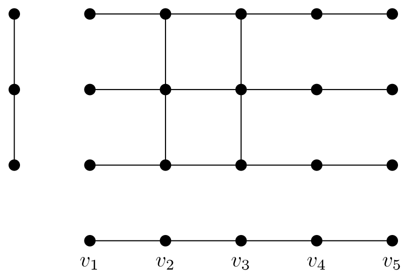

Fig. 1. $P_5(U) \sqcap P_3$ where $U = \{v_2, v_3\}$.

1.1. Preliminaries

We consider only finite, simple, and undirected graphs. Given sets $A, B \in V(G)$, we say that A *dominates* B if each vertex of B is adjacent to some vertex of A. A *dominating set of* G is a set $S \subset V(G)$ that dominates $V(G) - S$. The *domination number of* G, denoted $\gamma(G)$, is the minimum cardinality of a dominating set of G. Given a dominating set D of G, we say that a set $A \subset V(G)$ is *dominated externally by D* if $A \cap D = \emptyset$.

We focus on two different graph products in this paper. Given graphs $G_1, \ldots, G_n$ and vertex subsets $U_i \subseteq V(G_i)$ for $1 \le i \le n-1$, the *generalized hierarchical product*, denoted by $G = G_1(U_1) \sqcap \cdots \sqcap G_{n-1}(U_{n-1}) \sqcap G_n$ and defined in [4], is the graph with vertex set $V(G_1) \times \cdots \times V(G_n)$ and adjacencies

$$(x_1, \ldots, x_{n-1}, x_n) \sim \begin{cases} (y_1, x_2, \ldots, x_n) & \text{if } x_1y_1 \in E(G_1) \\ (x_1, y_2, x_3, \ldots, x_n) & \text{if } x_2y_2 \in E(G_2) \text{ and } x_1 \in U_1 \\ \vdots & \vdots \\ (x_1, \ldots, x_{n-1}, y_n) & \text{if } x_ny_n \in E(G_n) \text{ and } x_i \in U_i,\ 1 \le i \le n-1. \end{cases}$$

For each $i \in \{1, \ldots, n-1\}$, U_i in the above definition is referred to as the *root set of* G_i. L. Barrière et al. [3] first defined the "hierarchical product" to mean the graph $G_1(U_1) \sqcap \cdots \sqcap G_{n-1}(U_{n-1}) \sqcap G_n$ when $|U_i| = 1$ for $1 \le i \le n-1$. In this paper, we will refer to a "generalized hierarchical product" as simply a "hierarchical product". That is, we say a graph $G_1(U_1) \sqcap \cdots \sqcap G_{n-1}(U_{n-1}) \sqcap G_n$ is a hierarchical product, regardless of the cardinality of each U_i. In this paper, we will only study the hierarchical product of two graphs, $G(U) \sqcap H$ (see Fig. 1).

The *Cartesian product* of graphs G and H, denoted $G \Box H$, is the graph whose vertex set is $V(G) \times V(H)$ whereby two vertices (u_1, u_2) and (v_1, v_2) are adjacent if $u_1v_1 \in E(G)$ and $u_2 = v_2$ or $u_1 = v_1$ and $u_2v_2 \in E(H)$. In fact, $G \Box H \cong G(U) \sqcap H$ when $U = V(G)$. Given $G * H$ where $*$ represents either the hierarchical product or the Cartesian product, and some fixed $y \in V(H)$, we define the *G^y-layer* to be the graph induced by $\{(x, y) \in G * H \mid x \in V(G)\}$. If we are given a set $A \subset V(G)$, we refer to the set $\{(x, y) \in G * H \mid x \in A\}$ as the *A^y-cell*. There is a projection map $p_G : V(G * H) \to G$ defined as $p_G(x, y) = x$. Similarly, we define $p_H : V(G * H) \to H$ as $p_H(x, y) = y$.

2. Domination in the hierarchical product

Anderson et al. [1] gave lower bounds for $\gamma(G(U) \sqcap H)$ when either G is a path or when G has domination number 3 or 4 and U is a specific subset of $V(G)$. Their results primarily depend on the ability to always partition $V(G)$ in the following way.

Lemma 1 (*[1]*). *Let G be any graph with $\gamma(G) = k > 1$. There exists a partition $A_1, \ldots, A_k$ of $V(G)$ such that* (i) *A_i is a clique for $1 \le i \le k-1$* (ii), *for each $v \in A_k$, there exists $w \in A_i$ such that $vw \notin E(G)$ for $1 \le i \le k-1$, and* (iii) *for each $v \in A_i$ $1 \le i \le k-1$, there exists $w \in A_k$ such that $vw \notin E(G)$.*

We can now find a lower bound of $\gamma(G(U) \sqcap H)$ using the partition of $V(G)$ given in Lemma 1.

Theorem 2. *Let G and H be arbitrary graphs where $\gamma(G) > 1$. If $A_1, \ldots, A_{\gamma(G)}$ is a partition of $V(G)$ as in Lemma 1, then $\gamma(G(\bigcup_{i=1}^{\alpha} A_i) \sqcap H) \ge (2\gamma(G) - \alpha)\gamma(H)$ where $1 \le \alpha \le \gamma(G) - 1$.*

Proof. Let $\gamma(G) = k$, and let D be a dominating set of $G(\bigcup_{i=1}^{\alpha} A_i) \sqcap H$ where $1 \le \alpha \le k-1$. Define $\beta : V(H) \to \mathbb{N} \cup \{0\}$ by $\beta(y) = |D \cap G^y|$ for each $y \in V(H)$. Note that for all $y \in V(H)$, each A_i^y-cell for $\alpha + 1 \le i \le k$ is dominated from within its G^y-layer. We claim that this implies $\beta(y) \ge k - \alpha$ for each $y \in V(H)$. Indeed, if $\beta(y) < k - \alpha$ for some $y \in V(H)$, then we could choose one vertex x_i from each A_i for $1 \le i \le \alpha$ and $p_G(D \cap G^y) \cup \{x_i \mid 1 \le i \le \alpha\}$ would be a dominating set of G of cardinality at most $\gamma(G) - 1$, which is a contradiction. Thus, $\beta(y) \ge k - \alpha$ for each $y \in V(H)$.

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