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# On the determinant of the Laplacian matrix of a complex unit gain graph☆

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ABSTRACT

Let $G$ be a complex unit gain graph which is obtained from an undirected graph $\Gamma$ by assigning a complex unit $\varphi(v_iv_j)$ to each oriented edge $v_iv_j$ such that $\varphi(v_iv_j)\varphi(v_jv_i) = 1$ for all edges. The Laplacian matrix of $G$ is defined as $L(G) = D(G) - A(G)$, where $D(G)$ is the degree diagonal matrix of $\Gamma$ and $A(G) = (a_{ij})$ has $a_{ij} = \varphi(v_iv_j)$ if $v_i$ is adjacent to $v_j$ and $a_{ij} = 0$ otherwise. In this paper, we provide a combinatorial description of $\det(L(G))$ that generalizes that for the determinant of the Laplacian matrix of a signed graph.



## 1. Introduction

In the past few decades, researchers have extensively studied the adjacency, Laplacian, normalized Laplacian and signless Laplacian matrices of an undirected graph. Then there has been a growing study of matrices associated to a signed graph [9,14–17,30] and to an oriented graph [1,21]. Recently, researchers investigated matrices associated to a more general graph – a complex unit gain graph [20].

Let $\mathbb{T}$ be the circle group which is the multiplicative group of all complex numbers with absolute value 1. A $\mathbb{T}$-*gain graph* is arised from a simple graph with an orientation such that each orientation of an edge is given a complex unit of $\mathbb{T}$, called a gain, and the inverse of this complex unit assigned to the opposite orientation of such an edge. A $\mathbb{T}$-gain graph is also referred as a *complex unit gain graph*; see [20]. Let $\Gamma = (V, E)$ be the underlying graph of a $\mathbb{T}$-gain graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E$. $\vec{E} = \vec{E}(\Gamma)$ is defined to be the set of oriented edges of such a gain graph. For a $\mathbb{T}$-gain graph, denote by $e_{ij}$ the oriented edge from $v_i$ to $v_j$ and by $\varphi(e_{ij})$ the gain of $e_{ij}$. Hence a $\mathbb{T}$-gain graph is a triple $G = (\Gamma, \mathbb{T}, \varphi)$ consisting of an underlying graph $\Gamma = (V, E)$, the circle group $\mathbb{T}$ and a function $\varphi : \vec{E}(\Gamma) \to \mathbb{T}$ (called the gain function), such that $\varphi(e_{ij}) = \varphi(e_{ji})^{-1}$; see [20]. For simplicity, we sometimes write $G = (\Gamma, \varphi)$ for a $\mathbb{T}$-gain graph. For more properties of $\mathbb{T}$-gain graphs, one can see for example [3,12,23–29]. Note that the definition of a weighted directed graph by Bapat et al. [5] is same as a $\mathbb{T}$-gain graph.

Let $G = (\Gamma, \varphi)$ be a $\mathbb{T}$-gain graph with vertex set $\{v_1, v_2, \ldots, v_n\}$. The *adjacency matrix* $A(G) = (a_{ij})_{n\times n}$ of $G$ is defined by

$$a_{ij} = \begin{cases} \varphi(e_{ij}), & \text{if } v_i \text{ is adjacent to } v_j; \\ 0, & \text{otherwise.} \end{cases}$$

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One can see that if $v_i$ is adjacent to $v_j$, then $a_{ij} = \varphi(e_{ij}) = (\varphi(e_{ji}))^{-1} = \bar{a}_{ji}$, the conjugate of $a_{ji}$. Denote by $D(G) = diag\{d(v_1), d(v_2), \dots, d(v_2)\}$ the degree diagonal matrix of the underlying graph $\Gamma$. The *Laplacian matrix* $L(G) = (l_{ij})_{n\times n}$ of $G$ is defined as $L(G) = D(G) - A(G)$. Therefore, both of $A(G)$ and $L(G)$ are Hermitian.

Let $G$ be a $\mathbb{T}$-gain graph with Laplacian matrix $L(G)$. Bapat et al. [5] and Reff [20] independently point out that the definition of $L(G)$ coincides with the Laplacian matrix of the underlying graph of $\Gamma$ if $G$ has gain 1; $L(G)$ coincides with the signless Laplacian matrix of $\Gamma$ if $G$ has gain $-1$; and $L(G)$ coincides with the Laplacian matrix of a signed graph (a signed graph is also named by Bapat et al. as a mixed graph; see for example [4,5]) if $G$ has gains $\{1, -1\}$.

The graph obtained from a simple undirected graph by assigning an orientation to each of its edges is named as the oriented graph, denoted by $\overrightarrow{G}$. The *skew adjacency matrix* $A(\overrightarrow{G}) = (a_{ij})$ related to an oriented graph $\overrightarrow{G}$ is defined as $a_{ij} = -a_{ji} = 1$ if there exists an edge with tail $v_i$ and head $v_j$; and $a_{ij} = 0$ otherwise. The *skew Laplacian matrix* of $\overrightarrow{G}$ is defined as $L(\overrightarrow{G}) = D(\overrightarrow{G}) - A(\overrightarrow{G})$; see [2], where $D(\overrightarrow{G})$ denotes the degree diagonal matrix of $\overrightarrow{G}$. Unlike the adjacency matrix and the Laplacian matrix of an undirected graph, there has been little research on the skew-adjacency matrix $A(\overrightarrow{G})$ and the skew Laplacian matrix $L(\overrightarrow{G})$ of an oriented graph $\overrightarrow{G}$, except that in enumeration of perfect matchings of a graph, see [18] and references therein, where the square of the number of perfect matchings of a graph $\overrightarrow{G}$ with a Pfaffian orientation is the determinant of the skew-adjacency matrix $A(\overrightarrow{G})$. Recently, researchers investigated the spectral properties of matrices associated to an oriented graph; see [1,2,7,10,11,18,19,21,22]. By the definition of the adjacency matrix of a $\mathbb{T}$-gain graph, we can see that the adjacency matrix $A = (a_{ij})_{n\times n}$ of a graph $\mathbb{T}$ with gain set $\{\mathbf{i}, -\mathbf{i}\}$ can be considered the *skew adjacency matrix* of an oriented graph multiplied by the complex number $\mathbf{i}$, that is, $a_{ij} = -a_{ji} = \mathbf{i}$ if there exists an edge from $v_i$ to $v_j$; and $a_{ij} = 0$ otherwise. Therefore, the Laplacian matrix of the graph $\mathbb{T}$ with gain set $\{\mathbf{i}, -\mathbf{i}\}$ can be viewed as another version of the Laplacian matrix of oriented graphs, that is, $L(\overrightarrow{G}) = D(\overrightarrow{G}) - \mathbf{i}A(\overrightarrow{G})$.

Therefore, most classical graph matrices, including Laplacian, normalized Laplacian, signless Laplacian matrices of a graph, and the Laplacian matrix of an oriented graph can be viewed as a special case of the Laplacian matrix of a $\mathbb{T}$-gain graph.

The classical Matrix Tree Theorem in its simplest form [6, pp.219] gives a combinatorial characterization of a minor of the Laplacian matrix of a graph in terms of spanning trees of the underlying graph. Then the Matrix Tree Theorem for signed graphs is given by Zaslavsky [23, Theorem 8A.4] and a combinatorial proof of the all minors matrix tree theorem is given by Chaiken [8]. In this paper, we provide a combinatorial description the determinant of the Laplacian matrix of an arbitrary $\mathbb{T}$-gain graph, which is a generalization for the determinant of the Laplacian matrix of a signed graph.

## 2. The determinant of a complex unit gain graph

Throughout this paper, all $\mathbb{T}$-gain graphs have simple underlying graphs, *i.e.*, without loops and multi-edges, $\bar{a}$ denotes the *conjugate* of the complex number $a$ and $A^*$ denotes the *Hermitian transpose* of the complex matrix $A$. Given a $\mathbb{T}$-gain graph $G$, a maximal connected subgraph of $G$ is called a *component* of $G$. For convenience, in terms of defining subgraph and degree of a $\mathbb{T}$-gain graph, we focus only on its underlying graph. Certainly, each subgraph of a gain graph is also referred as a gain graph and preserves the gain of each edge, even if we do not state it specifically.

We first need to introduce the notion of the vertex–edge incidence matrix of a $\mathbb{T}$-gain graph, which was introduced in [29] for more general gain graphs. Let $G = (\Gamma, \varphi)$ be a $\mathbb{T}$-gain graph with vertex set $\Gamma(V) = \{v_1, v_2, \dots, v_n\}$ and edge set $\Gamma(E) = \{e_1, e_2, \dots, e_m\}$. Then the vertex–edge *incidence matrix* $M(G) = (m_{ij})_{n\times m}$ of $G$ is defined by

$$m_{v_i e} = \begin{cases} 1, & \text{if } e = e_{ij} \in \vec{E} \text{ for some vertex } v_j; \\ -\varphi(e_{ji}), & \text{if } e = e_{ji} \in \vec{E} \text{ for some vertex } v_j; \\ 0, & \text{otherwise.} \end{cases}$$

This definition can be considered as a particular incidence matrix related to a $\mathbb{T}$-gain graph defined by Reff [20]. From Lemma 3.1 in [20] $L(G) = M(G)M(G)^*$, then $L(G)$ is positive semi-definite and has a nonnegative determinant.

A connected $\mathbb{T}$-gain graph containing no cycles is called a $\mathbb{T}$-gain tree [20]. Since a $\mathbb{T}$-gain tree of order $n$ contains exactly $n - 1$ edges, its vertex–edge incidence matrix is an $n \times (n - 1)$ Hermitian matrix. We begin with the following result, which is a consequence of Corollary 3.4 in [20] or of Theorem 2.1 in [27].

**Lemma 2.1.** *Let $T$ be an arbitrary $\mathbb{T}$-gain tree with Laplacian matrix $L(T)$. Then*

$$\det(L(T)) = 0.$$

Let $C = v_1 e_{1,2} v_2 \cdots v_{s-1} e_{s-1,s} v_s (= v_1)$ be a cycle with $s(\geq 3)$ edges, where $v_j$ adjacent to $v_{j+1}$ for $j = 1, 2 \cdots, s - 1$ and $v_1$ incident to $v_s$. The gain of $C$, denoted by $\varphi(C)$, is defined as

$$\varphi(C) = \varphi(e_{1,2})\varphi(e_{2,3}) \cdots \varphi(e_{s-1,s})\varphi(e_{s,1}).$$

By the definition of the Laplacian matrix of a $\mathbb{T}$-gain graph, we have $l_{i,j} = -\varphi(e_{i,j})$ whenever $v_i$ adjacent to $v_j$, then $\varphi(C)$ can be defined in terms of the entries of its Laplacian matrix as

$$\varphi(C) = (-1)^s l_{1,2} l_{2,3} \cdots l_{s-1,s} l_{s,1}.$$

The following result shows that the determinant of the Laplacian matrix of a $\mathbb{T}$-gain cycle is determined by its gain.