Discrete Mathematics 341 (2018) 104–108

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Note

Degree Ramsey numbers for even cycles

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ARTICLE INFO

Article history:
Received 12 October 2016
Received in revised form 12 May 2017
Accepted 10 August 2017
Available online 11 September 2017

Keywords:
Ramsey number
Degree Ramsey number
Even cycle
Generalized polygon

ABSTRACT

Let $H \xrightarrow{s} G$ denote that any s-coloring of $E(H)$ contains a monochromatic G. The degree Ramsey number of a graph G, denoted by $R_\Delta(G, s)$, is $\min\{\Delta(H) : H \xrightarrow{s} G\}$. We consider degree Ramsey numbers where G is a fixed even cycle. Kinnersley, Milans, and West showed that $R_\Delta(C_{2k}, s) \geq 2s$, and Kang and Perarnau showed that $R_\Delta(C_4, s) = \Theta(s^2)$. Our main result is that $R_\Delta(C_6, s) = \Theta(s^{3/2})$ and $R_\Delta(C_{10}, s) = \Theta(s^{5/4})$. Additionally, we substantially improve the lower bound for $R_\Delta(C_{2k}, s)$ for general k.

1. Introduction

Theorems in Ramsey theory state that if a structure is in some suitable sense "large enough", then it must contain a fixed substructure. The classical *Ramsey number* of a graph G, denoted by $R(G, s)$, is the smallest n such that $K_n \xrightarrow{s} G$, where $H \xrightarrow{s} G$ denotes that any s-coloring of the edges of H produces a monochromatic subgraph isomorphic to G. Classical Ramsey numbers may be thought of in a more general setting, as just one type of parameter Ramsey number. Note that the classical Ramsey number of G is $\min\{|V(H)| : H \xrightarrow{s} G\}$. For any monotone graph parameter ρ, one may define the *ρ-Ramsey number* of G, denoted by $R_\rho(G, s)$, to be

$$\min\{\rho(H) : H \xrightarrow{s} G\}.$$

This generalizes the classical Ramsey number as $R_\rho(G, s)$ is the Ramsey number for G when $\rho(H)$ denotes the number of vertices in H. The study of parameter Ramsey numbers dates back to the 1970s [7]. Since then, many researchers have studied this quantity when $\rho(H)$ is the clique number of H [12,23,25] (giving way to the study of Folkman numbers), when $\rho(H)$ is the number of edges in H [3,4,9–11,13,27] (now called the *size Ramsey number*), when $\rho(H) = \chi(H)$ [7,28,29] or when it is the circular chromatic number [15]. In this note we are interested in the *degree Ramsey number*, which is when $\rho(H) = \Delta(H)$.

Burr, Erdős, and Lovász [7] showed that $R_\Delta(K_n, s) = R(K_n, s) - 1$. Kinnersley, Milans, and West [18] and Jiang, Milans, and West [16] proved several theorems regarding the degree Ramsey numbers of trees and cycles. Horn, Milans, and Rödl showed that the family of closed blowups of trees is R_Δ-bounded (that their degree Ramsey number is bounded by a function of the maximum degree of the graph and s). The main open question in this area is whether the set of all graphs is R_Δ-bounded (see [8]).

The main result of this note is to determine the order of magnitude of $R_\Delta(C_6, s)$ and $R_\Delta(C_{10}, s)$. Kang and Perarnau [17] showed that $R_\Delta(C_4, s) = \Theta(s^2)$. For general k, the best lower bound that is true for all values $k \geq 2$ on $R_\Delta(C_{2k}, s)$ is by Kinnersley, Milans, and West [18] who show $R_\Delta(C_{2k}, s) \geq 2s$. We substantially improve this lower bound in Theorem 1.3.

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http://dx.doi.org/10.1016/j.disc.2017.08.016

As the determination of Ramsey numbers for C_{2k} is closely related to the Turán number for C_{2k}, one may find it natural that the order of magnitude for $R_\Delta(C_{2k}, s)$ should be able to be determined for $k \in \{2, 3, 5\}$ but in no other cases. This is also the current state of affairs for the Turán numbers $\mathrm{ex}(n, C_{2k})$ as well as for the classical Ramsey numbers, where Li and Lih [22] showed that $R(C_{2k}, s) = \Theta\left(s^{k/(k-1)}\right)$ for $k \in \{2, 3, 5\}$.

Before we state our theorems, we need some preliminary definitions. For graphs H and G, we say that G *contains* H if H is a (not necessarily induced) subgraph of G. For graphs F and G, a *locally injective homomorphism* from F to G is a graph homomorphism $\phi : V(F) \to V(G)$ such that for every $v \in V(F)$, the restriction of ϕ to the neighborhood of v is injective. Let $\mathcal{L}_F$ denote the family of all graphs H such that there is a locally injective homomorphism from F to H. We say that a graph is $\mathcal{L}_F$-free if it does not contain any $H \in \mathcal{L}_F$. We now state our main theorem.

Theorem 1.1. *$R_\Delta(C_6, s) = \Theta(s^{3/2})$ and $R_\Delta(C_{10}, s) = \Theta(s^{5/4})$.*

To prove this theorem, we first show that the complete graph can be partitioned "efficiently" into graphs coming from the generalized quadrangle and generalized hexagon. Then we use the following general theorem, which is implicit in the work of Kang and Perarnau [17].

Theorem 1.2 (*Kang–Perarnau [17]*). *Let F be a graph with at least one cycle and $\epsilon > 0$ be fixed and let G be a graph of maximum degree Δ. Let f be a monotone non-decreasing function. If the edges of K_n can be partitioned into $f(n)O\left(n^{1-\epsilon}\right)$ $\mathcal{L}_F$-free graphs, then G can be partitioned into $f(200\Delta)O\left(\Delta^{1-\epsilon}\right)$ graphs which are F-free.*

They did not state their theorem in this way, and for completeness we sketch its proof in Section 2. Stating it in this general way allows us to improve the result of Kinnersley, Milans, and West [18].

Theorem 1.3. *Let k be fixed with $k \geq 2$ and $\delta = 0$ if k is odd and $\delta = 1$ if k is even. Then*

$$R_\Delta(C_{2k}, s) = \Omega\left(\left(\frac{s}{\log s}\right)^{1+\frac{2}{3k-5+\delta}}\right).$$

The general nature of Theorem 1.2 also allows one to read off the following corollary.

Corollary 1.4. *For $a > (b-1)!$, $R_\Delta(K_{a,b}, s) = \Theta(s^b)$.*

We prove our main theorem and Corollary 1.4 in Section 3. We sketch the proof of Theorem 1.2 in Section 2 and use it to prove Theorem 1.3 in Section 4.

2. Proof of Theorem 1.2

Throughout this section, assume that F is a graph with at least one cycle, that $\epsilon > 0$ is fixed, and that the edges of K_n can be partitioned into $O(n^{1-\epsilon})$ graphs which are $\mathcal{L}_F$-free. Call a vertex coloring of a graph a *proper rainbow coloring* if the coloring is proper, and the restriction of the coloring to any neighborhood is an injection (i.e. each vertex sees a rainbow).

Sketch of Proof of Theorem 1.2. To prove Theorem 1.2 we need the following lemma, which appears in [17]. Similar lemmas appear in [26] and [24].

Lemma 2.1. *Let G be a graph of sufficiently large maximum degree Δ and minimum degree $\delta \geq \log^2 \Delta$. Then there is a spanning subgraph H of G and a proper rainbow coloring $\chi : V(H) \to \{1, \ldots, 200\Delta\}$ such that for all $v \in V(G)$, $d_H(v) \geq \frac{1}{10} d_G(v)$.*

This lemma allows us to use the partition of K_n to partition a large piece of our graph G (viz H) into F-free subgraphs.

Proposition 2.2. *Let G be a graph of sufficiently large maximum degree Δ and minimum degree $\delta \geq \log^2 \Delta$. There exist $l = f(200\Delta)O(\Delta^{1-\epsilon})$ disjoint spanning subgraphs $H_1, \ldots, H_l$, all of which are F-free, such that for all $v \in V(G)$*

$$\sum_{i=1}^{l} d_{H_i}(v) \geq \frac{1}{10} d_G(v).$$

Proof. By assumption, the edge set of the complete graph $K_{200\Delta}$ on vertex set $\{1, \ldots, 200\Delta\}$ can be partitioned into $l = f(200\Delta)O(\Delta^{1-\epsilon})$ graphs which are $\mathcal{L}_F$-free. Denote these graphs by $G_1, \ldots, G_l$. Let H be the spanning subgraph of G with coloring χ from Lemma 2.1. Recall that χ is a proper rainbow coloring using at most 200Δ colors and that $d_H(v) \geq \frac{1}{10} d_G(v)$ for all v. For $1 \leq i \leq l$, define graphs H_i which are subgraphs of H by $V(H_i) = V(G)$ and $uv \in E(H_i)$ if and only if

$$\chi(u)\chi(v) \in E(G_i) \text{ and } uv \in E(H).$$

Since $G_1, \ldots, G_l$ is a partition of $E(K_{200\Delta})$, we have that $H_1, \ldots, H_l$ is a partition of H, and thus the minimum degree condition is satisfied. To see that each H_i is F-free, we claim that for each i, χ is a locally injective homomorphism from H_i to G_i. To see this, note that the definition of $E(H_i)$ guarantees that χ is a homomorphism from H_i to G_i, and χ being a rainbow coloring implies that the homomorphism is locally injective. Since G_i is $\mathcal{L}_F$-free, we have that H_i is F-free. ■