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More odd graph theory from another point of view

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ABSTRACT

The Kneser graph $K(n, k)$ has as vertices all k-element subsets of $[n] = \{1, 2, \ldots, n\}$ and an edge between any two vertices that are disjoint. If $n = 2k+1$, then $K(n, k)$ is called an odd graph. Let $n > 4$ and $1 < k < \frac{n}{2}$. In the present paper, we show that if the Kneser graph $K(n, k)$ is of even order where n is an odd integer or both of the integers n, k are even, then $K(n, k)$ is a vertex-transitive non Cayley graph. Although, these are special cases of Godsil (1980), unlike his proof that uses some very deep group-theoretical facts, ours uses no heavy group-theoretic facts. We obtain our results by using some rather elementary facts of number theory and group theory. We show that 'almost all' odd graphs are of even order, and consequently are vertex-transitive non Cayley graph. Finally, we show that If $k > 4$ is an even integer such that k is not of the form $k = 2^t$ for some $t > 2$, then the line graph of the odd graph O_{k+1} is a vertex-transitive non Cayley graph.

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1. Introduction and preliminaries

In this paper, a graph $\Gamma = (V, E)$ is considered as a finite, undirected, connected graph, without loops or multiple edges, where $V = V(\Gamma)$ is the vertex-set and $E = E(\Gamma)$ is the edge-set. For every terminology and notation not defined here, we follow [2,7,8,11].

The study of vertex-transitive graphs has a long and rich history in discrete mathematics. Prominent examples of vertex-transitive graphs are Cayley graphs which are important in both theory as well as applications. Vertex-transitive graphs that are not Cayley graphs, for which we use the abbreviation VTNCG, have been an object of a systematic study since 1980 [3,9]. In trying to recognize whether or not a vertex-transitive graph is a Cayley graph, we are left with the problem of determining whether the automorphism group contains a regular subgroup [2]. The reference [1] is an excellent source for studying graphs that are VTNCG.

Let $n > 4$ be an integer and $1 < k < \frac{n}{2}$. The Kneser graph $K(n, k)$ is the graph with the k-element subsets of $[n] = \{1, 2, \ldots, n\}$ as vertices, where two such vertices are adjacent if and only if they are disjoint. If $n = 2k + 1$, then the graph $K(2k + 1, k)$ is called an odd graph and is denoted by O_{k+1}. There are several good reasons for studying these graphs. One is that the questions which arise are related to problems in other areas of combinatorics, such as combinatorial set theory, coding theory, and design theory. A second reason is that the study of odd graphs tends to highlight the strengths and weaknesses of the techniques currently available in graph theory, and that many interesting problems and conjectures are encountered [3,4].

Amongst the various interesting properties of the Kneser graph $K(n, k)$, we interested in the automorphism group of it and we want to see how it acts on its vertex set. If $\theta \in Sym([n])$, then

$f_\theta : V(K(n, k)) \longrightarrow V(K(n, k)), f_\theta(\{x_1, \ldots, x_k\}) = \{\theta(x_1), \ldots, \theta(x_k)\}$

is an automorphism of $K(n, k)$ and the mapping $\psi : Sym([n]) \longrightarrow Aut(K(n, k))$, defined by the rule $\psi(\theta) = f_\theta$ is an injection. In fact, $Aut(K(n, k)) = \{f_\theta | \theta \in Sym([n])\} \cong Sym([n])$ [8], and for this reason we identify f_θ with θ when f_θ is an automorphism

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of $K(n, k)$, and in such a situation we write θ instead of f_θ. It is an easy task to show that the Kneser graph $K(n, k)$ is a vertex-transitive graph [8].

In 1979 Biggs [3] asked whether there are many values of k for which the odd graphs O_{k+1} are Cayley graphs. In 1980 Godsil [9] proved (by using some very deep group-theoretical facts of group theory [5,10]) that for 'almost all' values of k, the Kneser graph $K(n, k)$ is a *VTNCG*. In the present paper, we show that if the Kneser graph $K(n, k)$ is of even order where n is an odd integer or both of the integers n, k are even, then $K(n, k)$ is a *VTNCG*. We call the odd graph O_{k+1} an even–odd graph when its order is an even integer. We show that 'almost all' odd graphs are even–odd graphs, and consequently 'almost all' odd graphs are *VTNCG*. We obtain our results, by using some rather elementary facts of number theory and group theory.

Finally, we show that if $k > 4$ be an even integer and k is not of the form $k = 2^t$ for some $t > 2$, then the line graph of the odd graph O_{k+1} is a *VTNCG*.

2. Main results

Theorem 2.1. *Let n, k are integers, $n > 4$, $2 \leq k < \frac{n}{2}$ and $\binom{n}{k}$ is an even integer. Then the Kneser graph $K(n, k)$ is a vertex transitive non Cayley graph if one of the following conditions holds:*

(I) *n is an odd integer;*

(II) *n and k are even integers.*

Proof. We know that the Kneser graph $K(n, k)$ is a vertex-transitive graph (for every positive integer n) [8], hence it is sufficient to show that it is a non Cayley graph.

On the contrary, we assume that the Kneser graph $K(n, k)$ is a Cayley graph. Then the group $Aut(K(n, k)) = Sym([n])$, $[n] = \{1, 2, \ldots, n\}$ has a subgroup R, such that R acts regularly on the vertex-set of $K(n, k)$. In particular, the order of R is $\binom{n}{k}$, and since (by assumption) this number is an even integer, then 2 divides $|R|$. Therefore, by the Cauchy's theorem the group R has an element θ of order 2. We know that each element of $Sym([n])$ has a unique factorization into disjoint cycles of $Sym([n])$, hence we can write $\theta = \rho_1\rho_2\ldots\rho_h$, where each ρ_i is a cycle of $Sym([n])$ and $\rho_i \cap \rho_j = \varnothing$ when $i \neq j$. We also know that if $\theta = \rho_1\rho_2\ldots\rho_h$, where each ρ_i is a cycle of $Sym([n])$ and $\rho_i \cap \rho_j = \varnothing$, then the order of the permutation θ is the least common multiple of the integers, $|\rho_1|$, $|\rho_2|$, ..., $|\rho_h|$. Since θ is of order 2, then the order of each ρ_i is 2 or 1, say, $|\rho_i| \in \{1, 2\}$. In other words, each ρ_i is a transposition or a cycle of length 1. Let $\theta = \tau_1\tau_2\ldots\tau_a(i_1)(i_2)\ldots(i_b)$, where each τ_r is a transposition and each $i_s \in [n]$. We now argue the cases (I) and (II).

(I) Let $n = 2m + 1$, $m > 1$. Therefore, we have $2a + b = n = 2m + 1$, where b is an odd integer, and hence it is non-zero. Since b is a positive odd integer, then $b - 1$ is an even integer. we let $d = \frac{b-1}{2}$, so that d is a non-negative integer, $d < b$ and $m = a + d$. Let $\tau_r = (x_r y_r)$, $1 \leq r \leq a$, where $x_r, y_r \in [n]$. Now, there are two cases:

(i) $2a \leq k$, (ii) $2a > k$.

(ii) Suppose $2a \leq k$. Then there is some integer t such that $2a + t = k$, and since $2a + b = 2m + 1$, then $t \leq b$. Thus, for transpositions $\tau_1, \tau_2, \ldots, \tau_a$ and cycles $(i_1), \ldots, (i_t)$ of the cycle factorization of θ, the set $v = \{x_1, y_1, \ldots, x_a, y_a, i_1, i_2, \ldots, i_t\}$ is a k subset of the set $[n]$, and thus it is a vertex of the Kneser graph $K(n, k)$. Therefore, we have;

$$\theta(v) = \{\theta(x_1), \theta(y_1), \ldots, \theta(x_a), \theta(y_a), \theta(i_1), \ldots, \theta(i_t)\} = \{y_1, x_1, \ldots, y_a, x_a, i_1, i_2, \ldots, i_t\} = v.$$

(iii) Suppose $2a > k$. Then there is some integer c such that $2c \leq k$ and $2(c + 1) > k$. If $2c = k$, then we take the vertex $v = \{x_1, y_1, \ldots, x_c, y_c\}$, and hence we have;

$\theta(v) = \{\theta(x_1), \theta(y_1), \ldots, \theta(x_c), \theta(y_c)\} = \{y_1, x_1, \ldots, y_c, x_c\} = v$.

We now assume $2c < k$, then $2c + 1 = k$. Since $b \geq 1$, then for transpositions $\tau_1, \tau_2, \ldots, \tau_c$ and cycle (i_1) of the cycle factorization of θ, the set $v = \{x_1, y_1, \ldots, x_c, y_c, i_1\}$ is a k-subset of the set $[n]$, and therefore it is a vertex of the Kneser graph $K(n, k)$. Thus, we have;

$\theta(v) = \{\theta(x_1), \theta(y_1), \ldots, \theta(x_c), \theta(y_c), \theta(i_1)\} = \{y_1, x_1, \ldots, y_c, x_c, i_1\} = v$.

(II) Let $n = 2m$, $m > 2$ and $k = 2e$, $0 < 2e < m$. Since $\theta = \tau_1\tau_2\ldots\tau_a(i_1)(i_2)\ldots(i_b)$, $\tau_r = (x_r y_r)$, $1 \leq r \leq a$, where $x_r, y_r \in [n]$, then we have $2a + b = n = 2m$, where b is an even integer. We now consider the following cases.

If $a < e$, then $2a < 2e$, and hence there is some integer t such that $2a + t = 2e = k$. Since $2a + b = n > 2k = 4e$, then $t < b$. Therefore, $v = \{x_1, y_1, \ldots, x_a, y_a, i_1, \ldots, i_t\}$ is a vertex of $K(n, k)$.

If $a \geq e$, then $v = \{x_1, y_1, \ldots, x_e, y_e\}$ is a vertex of $K(n, k)$.

On the other hand, we can see that in every case we have $\theta(v) = v$.

From the above argument, it follows that θ fixes a vertex of the Kneser graph $K(n, k)$, which is a contradiction, because R acts regularly on the vertex-set of $K(n, k)$ and $\theta \in R$ is of order 2. This contradiction shows that the assertion of our theorem is true. □

Remark 2.2. Since $1 < k < \frac{n}{2}$, hence if in the case (i) of the above theorem, we add the condition,

'and assume that k is an even integer'

then we can construct two vertices v, w of $K(n, k)$ such that $\theta(v) = v$, $\theta(w) = w$ and $v \cap w = \varnothing$. In fact, if $k = 2l$, $l > 0$, and $2a \leq k = 2l$, we have no problem for constructing the vertices v and w. If $2a > k = 2l$, then we can construct,

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