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Properties of minimally t-tough graphs

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ABSTRACT

A graph G is minimally t-tough if the toughness of G is t and the deletion of any edge from G decreases the toughness. Kriesell conjectured that for every minimally 1-tough graph the minimum degree $\delta(G) = 2$. We show that in every minimally 1-tough graph $\delta(G) \le \frac{n}{3} + 1$. We also prove that every minimally 1-tough, claw-free graph is a cycle. On the other hand, we show that for every positive rational number t any graph can be embedded as an induced subgraph into a minimally t-tough graph.

1. Introduction

All graphs considered in this paper are finite, simple and undirected. Let $d(v)$ denote the degree of a vertex v, $\omega(G)$ denote the number of components, $\alpha(G)$ denote the independence number and $\delta(G)$ denote the minimum degree of a graph G.

Definition 1.1. A graph G is k-connected, if it has at least $k+1$ vertices and remains connected whenever fewer than k vertices are removed. The connectivity of G, denoted by $\kappa(G)$, is the largest k for which G is k-connected.

The more edges a graph has, the larger its connectivity can be, so the graphs, which are k-connected and have the fewest edges for this property, may be interesting.

Definition 1.2. A graph G is minimally k-connected, if $\kappa(G) = k$ and $\kappa(G-e) < k$ for all $e \in E(G)$.

Clearly, all degrees of a k-connected graph have to be at least k. On the other hand, Mader proved that the minimum degree of every minimally k-connected graph is exactly k.

Theorem 1.3 (*Mader [6]*)**.** *Every minimally k-connected graph has a vertex of degree k.*

The notion of toughness was introduced by Chvátal [2] in 1973.

Definition 1.4. Let t be a positive real number. A graph G is called t-tough, if $\omega(G-S) \le |S|/t$ for any cutset S of G. The toughness of G, denoted by $\tau(G)$, is the largest t for which G is t-tough, taking $\tau(K_n) = \infty$ for all $n \ge 1$.

We say that a cutset $S \subseteq V(G)$ is a tough set if $\omega(G-S) = |S|/\tau(G)$.

We can define an analogue of minimally k-connected graphs for the notion of toughness.

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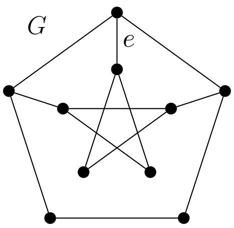

Fig. 1. A minimally 1-tough but not minimally 2-connected graph. The graph $G - e$ is still 2-connected.

Definition 1.5. A graph G is said to be minimally t-tough, if $\tau(G) = t$ and $\tau(G - e) < t$ for all $e \in E(G)$.

It follows directly from the definition that every t-tough graph is $2t$-connected, implying $\kappa(G) \geq 2\tau(G)$ for noncomplete graphs. Therefore, the minimum degree of any 1-tough graph is at least 2. Kriesell conjectured that the analogue of Mader's theorem holds for minimally 1-tough graphs.

Conjecture 1.6 (*Kriesell [4]*)**.** *Every minimally* 1*-tough graph has a vertex of degree* 2*.*

A 1-tough graph is always 2-connected, however, a minimally 1-tough graph is not necessarily minimally 2-connected (see Fig. 1), so Mader's theorem cannot be applied.

A natural approach to Kriesell's conjecture is to prove upper bounds on $\delta(G)$ for minimally 1-tough graphs. Kriesell's conjecture states that $\delta(G) \leq 2$, and the best known upper bound follows easily from Dirac's theorem, yielding $\delta(G) \leq n/2$. Our main result is an improvement on the current upper bound by a constant factor.

Theorem 1.7. *Every minimally 1-tough graph has a vertex of degree at most* $\frac{n}{3} + 1$*.*

Toughness is related to the existence of Hamiltonian cycles. If a graph contains a Hamiltonian cycle, then it is necessarily 1-tough. The converse is not true, a well-known counterexample is the Petersen graph. It is easy to see that every minimally 1-tough, Hamiltonian graph is a cycle, since after deleting an edge that is not contained by the Hamiltonian cycle, the resulting graph is still 1-tough.

Let us introduce a class of graphs that is frequently studied while dealing with problems related to Hamiltonian cycles.

Definition 1.8. The graph $K_{1,3}$ is called a claw. A graph is said to be claw-free, if it does not contain a claw as an induced subgraph.

Problems about connectivity in claw-free graphs can be handled more easily, since every vertex of a cutset is adjacent to at most two components. We give a complete characterization of minimally 1-tough, claw-free graphs.

Theorem 1.9. *If G is a minimally 1-tough, claw-free graph of order* $n \geq 4$*, then* $G = C_n$*.*

Thus we see that Kriesell's conjecture is true in a very strong sense if the graph is claw-free. Or equivalently, the family of minimally 1-tough, claw-free graphs is small. On the other hand, we show that in general the class of minimally 1-tough graphs is large.

Theorem 1.10. *For every positive rational number t, any graph can be embedded as an induced subgraph into a minimally t-tough graph.*

The paper is organized as follows. In Section 2 we prove Theorem 1.7 which is the main result of this paper. In Section 3 we prove Theorem 1.9 and in Section 4 we prove Theorem 1.10.

2. Proof of the main result

Here we prove that every minimally 1-tough graph has a vertex of degree at most $\frac{n}{3} + 1$. First we need a claim that has a key role in the proofs, then we continue with two lemmas.

Claim 2.1. *If G is a minimally* 1*-tough graph, then for every edge* $e \in E(G)$ *there exists a vertex set* $S = S(e) \subseteq V(G)$ *with*

$$\omega(G - S) = |S| \quad \text{and} \quad \omega\big((G - e) - S\big) = |S| + 1.$$

Proof. Let e be an arbitrary edge of G. Since G is minimally 1-tough, $\tau(G - e) < 1$, so there exists a cutset $S = S(e) \subseteq V(G - e) = V(G)$ in $G - e$ satisfying that $\omega\big((G - e) - S\big) > |S|$. On the other hand, $\tau(G) = 1$, so $\omega(G - S) \leq |S|$. This is only possible if e connects two components of $(G - e) - S$, which means $\omega\big((G - e) - S\big) = |S| + 1$ and $\omega(G - S) = |S|$. □