Discrete Mathematics 341 (2018) 277–285

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Constructions for orthogonal designs using signed group orthogonal designs

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ARTICLE INFO

Article history:
Received 3 October 2015
Received in revised form 19 August 2016
Accepted 28 August 2017
Available online 19 September 2017

Keywords:
Circulant matrix
Golay pair
Hadamard matrix
Orthogonal design
Signed group orthogonal design

ABSTRACT

Craigen introduced and studied signed group Hadamard matrices extensively and eventually provided an asymptotic existence result for Hadamard matrices. Following his lead, Ghaderpour introduced signed group orthogonal designs and showed an asymptotic existence result for orthogonal designs and consequently Hadamard matrices. In this paper, we construct some interesting families of orthogonal designs using signed group orthogonal designs to show the capability of signed group orthogonal designs in generation of different types of orthogonal designs.

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1. Preliminaries

A *Hadamard* matrix [7,16] is a square matrix with entries from $\{\pm 1\}$ whose rows are pairwise orthogonal. An *orthogonal design* (OD) [2,7,16] of order n and type $(c_1, \ldots, c_k)$, denoted by $OD(n;\ c_1, \ldots, c_k)$, is a square matrix X of order n with entries from $\{0, \pm x_1, \ldots, \pm x_k\}$ that satisfies

$$XX^{\mathrm{T}} = \Big(\sum_{j=1}^{k} c_j x_j^2\Big) I_n,$$

where the c_j's are positive integers, the x_j's are commuting variables, I_n is the identity matrix of order n, and X^{T} is the transpose of X. An OD with no zero entry is called a *full* OD. A Hadamard matrix can be obtained by equating all variables of a full OD to 1. The maximum number of variables in an OD of order $n = 2^a b$, b odd, is $\rho(n) = 8c + 2^d$, where $a = 4c + d$, $0 \le d < 4$. This number is called *Radon-Hurwitz number* [7, Chapter 1].

A *complex orthogonal design* (COD) [2,6,8] of order n and type $(c_1, \ldots, c_k)$, denoted by $COD(n;\ c_1, \ldots, c_k)$, is a square matrix X of order n with entries from $\{0, \pm x_1, \pm ix_1, \ldots, \pm x_k, \pm ix_k\}$ that satisfies

$$XX^{*} = \Big(\sum_{j=1}^{k} c_j x_j^2\Big) I_n,$$

where the c_j's are positive integers, the x_j's are commuting variables, and $*$ is the conjugate transpose.

Two matrices A and B of the same dimension are called *disjoint* [7,10,16] if the matrix computed via entrywise multiplication of A and B is a zero matrix. Pairwise disjoint matrices such that their sum has no zero entries are called *supplementary* [2,7].

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http://dx.doi.org/10.1016/j.disc.2017.08.039
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The *Kronecker product* [7,10] of two matrices $A = [a_{ij}]$ and B of orders $m \times n$ and $r \times s$, respectively, denoted by $A \otimes B$, is defined by

$$A \otimes B := \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & & & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix},$$

that is a matrix of order $mr \times ns$.

The *non-periodic autocorrelation function* [11] of a sequence $A = (x_1, \ldots, x_n)$ of commuting square complex matrices of order m, is defined by

$$N_A(j) := \begin{cases} \sum_{i=1}^{n-j} x_{i+j}x_i^* & \text{if } j = 0, 1, 2, \ldots, n-1, \\ 0 & j \geq n, \end{cases}$$

where $*$ is the conjugate transpose. A set $\{A_1, A_2, \ldots, A_\ell\}$ of sequences (not necessarily in the same length) is said to have zero autocorrelation if for all $j > 0$, $\sum_{k=1}^{\ell} N_{A_k}(j) = 0$. Sequences having zero autocorrelation are called *complementary* [7].

A pair $(A; B)$ of $\{\pm 1\}$-complementary sequences of length n is called a *Golay pair* of length n. A *Golay number* is a positive integer n such that there exists a Golay pair of length n. Similarly, a pair $(C; D)$ of $\{\pm 1, \pm i\}$-complementary sequences of length m is called a *complex Golay pair* of length m. A *complex Golay number* is a positive integer m such that there exists a complex Golay pair of length m [3,4,7].

A *signed group* S [2,3,8] is a group with a distinguished central element of order two. We denote the unit of a signed group by 1 and the distinguished central element of order two by -1. In every signed group, the set $\{1, -1\}$ is a normal subgroup, and the *order* of signed group S is the number of elements in the quotient group $S/\langle -1 \rangle$. Therefore, a signed group of order n is a group of order $2n$.

For instance, the trivial signed group $S_{\mathbb{R}} = \{1, -1\}$ is a signed group of order 1, the complex signed group $S_{\mathbb{C}} = \langle i : i^2 = -1 \rangle = \{\pm 1, \pm i\}$ is a signed group of order 2, the quaternion signed group $S_Q = \langle j, k : j^2 = k^2 = -1, jk = -kj \rangle = \{\pm 1, \pm j, \pm k, \pm jk\}$ is a signed group of order 4, and the set of all monomial $\{0, \pm 1\}$-matrices of order n, SP_n, forms a group of order $2^n n!$ and a signed group of order $2^{n-1}n!$. The distinguished central elements of $S_{\mathbb{R}}$, $S_{\mathbb{C}}$ and S_Q are all -1, and the distinguished central element of SP_n is $-I_n$, where I_n is the identity matrix of order n.

A signed group S' is called a *signed subgroup* [2,3] of a signed group S, denoted by $S' \leq S$, if S' is a subgroup of S, and the distinguished central elements of S' and S coincide. As an example, we have $S_{\mathbb{R}} \leq S_{\mathbb{C}} \leq S_Q$.

Let S be a signed group and $T \leq SP_n$. A *remrep* (*real monomial representation*) [2,3,5] of degree n is a map $\phi : S \to T$ such that for all $a, b \in S$, $\phi(ab) = \phi(a)\phi(b)$ and $\phi(-1) = -I_n$.

If R is a ring with unit 1_R, and S is a signed group with distinguished central element -1_S, then $R[S] := \left\{\sum_{i=1}^{n} s_i r_i : s_i \in \varrho,\ r_i \in R\right\}$ is a signed group ring [2,3], where ϱ is a set of coset representatives of S modulo $\langle -1_S \rangle$. The set ϱ is often referred to as a *transversal* of $\langle -1_S \rangle$ in S. For $s \in \varrho, r \in R$, we make the identification $-sr = s(-r)$. Addition is defined termwise, and multiplication is defined by linear extension. For instance, $s_1r_1(s_2r_2 + s_3r_3) = s_1s_2r_1r_2 + s_1s_3r_1r_3$, where $s_i \in \varrho, r_i \in R$ $i \in \{1, 2, 3\}$.

In this work, we choose $R = \mathbb{R}$. If $x \in \mathbb{R}[S]$, then $x = \sum_{i=1}^{n} s_i r_i$, where $s_i \in \varrho, r_i \in \mathbb{R}$, and we define the *conjugate* of x by $\bar{x} := \sum_{i=1}^{n} \bar{s}_i r_i = \sum_{i=1}^{n} s_i^{-1} r_i$. Clearly, the conjugate is an involution that is $\bar{\bar{x}} = x$ for all $x \in \mathbb{R}[S]$, and $\overline{xy} = \bar{y}\bar{x}$ for all $x, y \in \mathbb{R}[S]$. As some examples, for any $a, b \in \mathbb{R}$, we have $\overline{a + ib} = a + \bar{i}b = a + i^{-1}b = a - ib$, where $i \in S_{\mathbb{C}}$, and $\overline{ja + jkb} = j^{-1}a + (jk)^{-1}b = -ja - jkb$, where $j, k \in S_Q$.

A *circulant* matrix C [2,7,10] is a square matrix whose each row vector is rotated one element to the right with respect to the previous row vector, and we denote it by circ $(a_1, a_2, \ldots, a_n)$, where $(a_1, a_2, \ldots, a_n)$ is its first row. The circulant matrix C can be written as $C = a_1 I_n + \sum_{k=1}^{n-1} a_{k+1} U^k$, where $U = \text{circ}\,(0, 1, 0, \ldots, 0)$ (see [7, Chapter 4]). Therefore, any two circulant matrices of order n with commuting entries commute. If $C = \text{circ}\,(a_1, a_2, \ldots, a_n)$, then $C^* = \text{circ}\,(\bar{a}_1, \bar{a}_n, \ldots, \bar{a}_2)$, where $*$ is the conjugate transpose.

Suppose that $A = (a_1, a_2, \ldots, a_n)$ is a sequence whose nonzero entries are elements of a signed group S multiplied on the right by variables x_i's $(1 \leq i \leq k)$. We use $A_{\bar{R}}$ to denote a sequence whose elements are those of A, conjugated and in reverse order [4,9] that is $A_{\bar{R}} = (\bar{a}_n, \ldots, \bar{a}_2, \bar{a}_1)$.

A *signed group weighing matrix* (SW) [3,5] of order n and weight w *over* a signed group S, denoted by $SW(n, w, S)$, is a $(0, S)$-matrix (that is a matrix whose nonzero entries are in S) W such that $WW^* = wI_n$, where $*$ is the conjugate transpose. An SW over S with no zero entry $(w = n)$ is called a *signed group Hadamard matrix* (SH) over S [2,3], denoted by $SH(n, S)$. Note that the matrix operations for SWs and SHs are in the signed group ring $\mathbb{Z}[S]$.

Two square matrices A and B are called *amicable* if $AB^* = BA^*$, and they are called *anti-amicable* if $AB^* = -BA^*$, where $*$ is the conjugate transpose [2,7,8,16]. If the entries of A and B belong to a signed group ring, then the matrix operations are in the signed group ring as mentioned above.

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