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The minimum volume of subspace trades ☆

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ABSTRACT

A subspace bitrade of type $T_q(t, k, v)$ is a pair (T_0, T_1) of two disjoint nonempty collections of k-dimensional subspaces of a v-dimensional space V over the finite field of order q such that every t-dimensional subspace of V is covered by the same number of subspaces from T_0 and T_1. In a previous paper, the minimum cardinality of a subspace $T_q(t, t+1, v)$ bitrade was established. We generalize that result by showing that for admissible v, t, and k, the minimum cardinality of a subspace $T_q(t, k, v)$ bitrade does not depend on k. An example of a minimum bitrade is represented using generator matrices in the reduced echelon form. For $t = 1$, the uniqueness of a minimum bitrade is proved.

1. Introduction

Trades (see, e.g., [7]) are traditionally used for constructing and studying combinatorial designs. The relation of trades and designs can be briefly described as follows: if a design includes a trade subset, then this subset can be replaced by another (mate) trade to form a new design with the same parameters. This operation, also known as switching (see, e.g., [13]), is widely studied not only for designs but for many different kinds of combinatorial objects (latin squares and hypercubes, codes, etc.). In view of the growing interest to the q-ary (subspace) generalizations of designs in the last few years, the study of corresponding analogs of trades becomes actual. Subspace trades are already used in the construction of subspace designs [2].

In the current paper, we establish the minimum possible cardinality of a subspace $T_q(t, k, v)$ trade corresponding to the q-ary generalizations of $S(t, k, v)$ designs. The case $T_q(t, t+1, v)$ was solved in [5]. In [5], the equivalent language of null designs was used instead of trades, and the result for the subspace trades appears as a partial case of more general theory of trades (null designs) in ranked posets. In [11], the same partial result (on the minimum subspace trades with $k = t + 1$) was represented in a different general context, in terms of so-called clique trades in distance-regular graphs with regular systems of Delsarte cliques. (Regretfully, the authors of [11] did not refer [5] properly as they were not familiar with the theory of null designs. As one of them, the author of the current paper is also responsible for this inconvenience.) We will use the solution for $k = t+1$ to prove the general result for subspace trades. As in [11], the proof of the bound in the current paper exploits a weight distribution of the characteristic function of a bitrade (a bitrade, also known as a 2-way trade, is a pair of mate trades).

There is one essential difference between the $T_q(t, t+1, v)$ trades and the general case. In the first case, the characteristic function of a bitrade is an eigenfunction of the Grassmann graph, and the weight distribution with respect to a single vertex works well for establishing a lower bound on the number of nonzeros of the function. In the general case, the eigenspectrum of the bitrade characteristic function consists of several eigenvalues. To neutralize all components but one, we modify the

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technique and count the weight distribution with respect to a special completely regular set. The invariance of the considered distributions allows to prove the tight lower bound on the cardinality of a bitrade, but at the moment, does not allow to generalize the distance-regular properties of minimum $T_q(t, t+1, v)$ trades that were established in [11].

Another difference is that for $k = t + 1$ a minimum $T_q(t, k, v)$ trade is a Steiner trade, that is, every t-subspace is covered by at most one k-subspace from the trade (this follows from the explicit construction and the uniqueness of such a trade [4]). This is not necessarily the case in general; the known example of a minimum trade (see Section 3.3) is not Steiner if $k > t+1$. The minimum size of a Steiner subspace $T_q(t, k, v)$ trade, $k > t + 1$, remains unknown.

Many results on subspace designs can be treated as analogs (sometimes, essentially more complicated) of similar facts in the theory of ordinary designs, and our result is not an exception. The minimum volume 2^t of the ordinary trade was established in [6] and [8]. Our method can be considered as the development of the method of [6]; however, we use an alternative terminology of the theory of distance-regular graphs instead of the terminology of ranked posets, which was developed in [5] in this context. It would be interesting to find a subspace analog of the elegant inductive proof of the lower bound 2^t on the minimum trade volume found in [8].

In Section 2 we introduce the notation and the main theorem. Section 3 contains a proof of the result. In Section 4 we discuss some open problems, including the uniqueness of minimum subspace trades (we prove it for $t = 1$).

2. Notation and the main theorem

- q is a prime power; v is an integer, $v \geq 4$.
- $\mathbb{F}^v$—a v-dimensional space over the finite field $\mathbb{F} = \mathrm{GF}(q)$ of order q.
- $\mathcal{F}_i^v$—the set of all i-dimensional subspaces of $\mathbb{F}^v$, $i \in \{0, \dots, v\}$.
- $J_q(v, i)$—the *Grassmann graph* on the vertex set $\mathcal{F}_i^v$; two subspaces $X, Y \in \mathcal{F}_i^v$ are adjacent if $\dim(X \cap Y) = i - 1$.
- $d(X, Y)$—the natural graph distance between two vertices X and Y of the graph. The distance from a vertex to a set of vertices is defined in a usual way: $d(X, \mathcal{Y}) := \min_{Y \in \mathcal{Y}} d(X, Y)$.
- t, k—integers satisfying $0 \leq t < k < v - t$.
- A pair $(\mathcal{T}_0, \mathcal{T}_1)$ of disjoint nonempty multisubsets of $\mathcal{F}_k^v$ is called a $T_q(t, k, v)$ *subspace bitrade* if every subspace from $\mathcal{F}_t^v$ is covered by the same number of subspaces from $\mathcal{T}_0$ and $\mathcal{T}_1$ (the readers who are not interested in trades with repetitions can imply that $\mathcal{T}_0$ and $\mathcal{T}_1$ are ordinary sets, without multiplicities).
- We will refer to the value $|\mathcal{T}_0 \cup \mathcal{T}_1|$ as the *cardinality* of a bitrade $(\mathcal{T}_0, \mathcal{T}_1)$, while the value $|\mathcal{T}_0|$ is known as its *volume* (from the definition, it follows that $|\mathcal{T}_0| = |\mathcal{T}_1|$).

Our goal is to prove the following theorem.

Theorem 1. *Let the integers t, k, and v satisfy $0 \leq t < k < v - t$. The minimum possible cardinality $|\mathcal{T}_0 \cup \mathcal{T}_1|$ of a subspace bitrade $(\mathcal{T}_0, \mathcal{T}_1)$ of type $T_q(t, k, v)$ equals*

$$\prod_{i=0}^{t}(1 + q^i) = \sum_{j=0}^{t+1} q^{\frac{j(j-1)}{2}} \begin{bmatrix} t+1 \\ j \end{bmatrix}_q \tag{1}$$

where $\begin{bmatrix} i \\ j \end{bmatrix}_q := \frac{[i]_q[i-1]_q \dots [i-j+1]_q}{[1]_q[2]_q \dots [j]_q}$, $[r]_q := 1 + q + \dots + q^{r-1}$.

An example of a minimum bitrade is given in Section 3.3. Formula (1) is a known identity [15, Equation (1.87)]; in the proof, we will refer to its right part. Theorem 1 is the subspace analog of the similar result for the classical design trades (where the role of $\mathbb{F}^v$ is played by a set of v elements, the role of the i-dimensional subspaces is played by the i-subsets, and the formulas hold with $q = 1$) proved in [6,8].

The next group of definitions and notations concerns the space of real-valued functions on $\mathcal{F}_k^v$. Such functions will be denoted by lowercase Greek letters; the only exception is θ, which will always denote an eigenvalue.

- For a subset $\mathcal{S}$ of the vertex set $\mathcal{V}$ of a graph, we denote

  $$\mathcal{S}^{(i)} := \{Y \in \mathcal{V} \mid d(Y, \mathcal{S}) = i\}.$$

- A real-valued function φ on the set $\mathcal{V}$ of vertices of a simple graph is called an *eigenfunction* with *eigenvalue* θ if for all X from $\mathcal{V}$ it holds

  $$\sum_{Y \in \{X\}^{(1)}} \varphi(Y) = \theta \varphi(X).$$

- Two real-valued functions φ, ψ on $\mathcal{V}$ are *orthogonal*, $\varphi \perp \psi$, if $\sum_{X \in \mathcal{V}} \varphi(X)\psi(X) = 0$.
- $\bar{k} := \min(k, v - k)$—the diameter of $J_q(v, k)$
- $\theta_0, \theta_1, \dots, \theta_{\bar{k}}$—the eigenvalues of $J_q(v, k)$; we assume $\theta_0 > \theta_1 > \dots > \theta_{\bar{k}}$.
- $\Theta_0, \Theta_1, \dots, \Theta_{\bar{k}}$—the eigenspaces corresponding to the eigenvalues $\theta_0, \theta_1, \dots, \theta_{\bar{k}}$, respectively. So, $\Theta_0 + \Theta_1 + \dots + \Theta_{\bar{k}}$ is the space of all real-valued functions over $\mathcal{F}_k^v$.