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Chained permutations and alternating sign matrices—Inspired by three-person chess

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A B S T R A C T

We define and enumerate two new two-parameter permutation families, namely, placements of a maximum number of non-attacking rooks on k chained-together $n \times n$ chessboards, in either a circular or linear configuration. The linear case with $k = 1$ corresponds to standard permutations of n, and the circular case with $n = 4$ and $k = 6$ corresponds to a three-person chessboard. We give bijections of these rook placements to matrix form, one-line notation, and matchings on certain graphs. Finally, we define chained linear and circular alternating sign matrices, enumerate them for certain values of n and k, and give bijections to analogues of monotone triangles, square ice configurations, and fully-packed loop configurations.

1. Introduction

A typical enumeration problem given in an introductory combinatorics course is the following: How many ways are there to place m non-attacking rooks on an $n \times n$ chessboard? The solution is to first choose which m rows the rooks occupy, in $\binom{n}{m}$ ways, then the falling factorial $(n)_m := n(n-1)(n-2)\cdots(n-m+1)$ counts the number of ways to place the m rooks on those m rows. So there are $\binom{n}{m}(n)_m$ such rook placements. In the special case of placing the maximum number n of rooks on the $n \times n$ board, this reduces to $n!$.

One natural extension of this question is to change the rules for how the piece moves. For example, one may want to count non-attacking queen placements rather than rook placements; see [3,10,11]. A different extension of the question is to change the chessboard. The beautiful theory of *rook polynomials*, studied by Goldman, Joichi, and White in [5–9], discusses the generating function of the number of rook placements on any sub-board of the $n \times n$ board and shows when the generating function of two boards is equivalent.

This paper generalizes the theory of rook placements by considering a different kind of board, namely, a board created by chaining together multiple $n \times n$ chessboards in a particular way that we describe in Definition 2.1.

This work was inspired by the board game *three-person chess*. Though the game had been gathering dust in the fifth author's closet and the directions for game play had been lost, the board still inspired the following combinatorial question: **How many ways are there to place m non-attacking rooks on the three-person chessboard of** Fig. 1?

In this paper, we answer this question and generalize this result to a two-parameter family, namely, *maximum rook placements on k chained-together $n \times n$ boards* in either a *linear* or *circular* configuration. We highlight below our main results.

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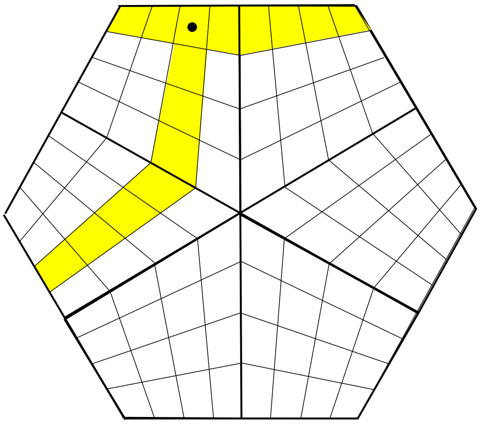

Fig. 1. A three-person chessboard; the dot represents a rook and the highlighted cells are the cells the rook is attacking. See Figs. 3 and 4 to see how this board transforms to $B^{\circ}_{4,6}$.

Our first main theorem, stated below, gives a formula for the number of non-attacking rook placements of m rooks in either of these families for any values of n and k. Let $B^{-}_{n,k}$ denote the *linear configuration* of k chained $n \times n$ chessboards and $B^{\circ}_{n,k}$ the *circular configuration*; see Definition 2.1. Also, see Definition 2.2 for the definition of $\mathfrak{C}_m(B)$.

Theorem 2.4. *The number of ways to place m non-attacking rooks on board* $B \in \{B^{-}_{n,k}, B^{\circ}_{n,k}\}$ *is*

$$\sum_{(a_1,\ldots,a_k)\in\mathfrak{C}_m(B)} \prod_{i=1}^{k} \binom{n-a_{i-1}}{a_i}(n)_{a_i}$$

where a_0 *is defined as follows:*

$$a_0 = \begin{cases} 0 & \text{if } B = B^{-}_{n,k} \\ a_k & \text{if } B = B^{\circ}_{n,k}. \end{cases}$$

We use this theorem to determine exact counts of placements of the *maximum* number of non-attacking rooks on each board.

Theorem 2.7. *The number of maximum rook placements on* $B^{-}_{n,k}$ *is given by:*

- *Case k even:* $(n!)^{\frac{k}{2}} \sum_{0\le j_1\le\cdots\le j_{\frac{k}{2}}\le n} \prod_{\ell=1}^{\frac{k}{2}} \binom{n-j_{\ell-1}}{n-j_\ell}\binom{n}{j_\ell}$,
- *Case k odd:* $(n!)^{\frac{k+1}{2}}$.

Theorem 2.10. *The number of maximum rook placements on* $B^{\circ}_{n,k}$ *is given by:*

- *Case k even:* $(n!)^{\frac{k}{2}} \sum_{j=0}^{n} \binom{n}{j}^{\frac{k}{2}}$,
- *Case k odd, n even:* $\left((n)_{\frac{n}{2}}\right)^{k}$,
- *Case k odd, n odd:* $k\left((n)_{\lceil\frac{n}{2}\rceil}\right)^{\lfloor\frac{k}{2}\rfloor}\left((n)_{\lfloor\frac{n}{2}\rfloor}\right)^{\lceil\frac{k}{2}\rceil}$.

We then shift from discussing rook placements to the study of *chained permutations*, which are equivalent to maximum rook placements on these boards. In Theorems 3.7 and 3.10, we transform chained permutations into forms analogous to the one-line notation and perfect matching form of standard permutations.

Finally, we define *chained alternating sign matrices* (Definition 4.1). In Proposition 4.6 **through** Corollary 4.14 we enumerate them for special values of n and k; in Theorems 4.18, 4.21 and 4.23, we transform them into forms analogous to monotone triangles, square ice configurations, and fully-packed loop configurations.

Our outline is as follows. In Section 2, we define the boards $B^{-}_{n,k}$ and $B^{\circ}_{n,k}$ and prove Theorems 2.4, 2.7 and 2.10 which enumerate non-attacking rook placements on these boards. In Section 3, we transform the maximum rook placements to chained permutations and prove Theorems 3.7 and 3.10 which give further bijections. In Section 4, we define chained alternating sign matrices, enumerate them in special cases, and prove the further bijections of Theorems 4.18, 4.21, and 4.23.