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Covering complete partite hypergraphs by monochromatic components

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ABSTRACT

A well-known special case of a conjecture attributed to Ryser (actually appeared in the thesis of Henderson (1971)) states that k-partite intersecting hypergraphs have transversals of at most $k-1$ vertices. An equivalent form of the conjecture in terms of coloring of complete graphs is formulated in Gyárfás (1977): if the edges of a complete graph K are colored with k colors then the vertex set of K can be covered by at most $k-1$ sets, each forming a connected graph in some color. It turned out that the analogue of the conjecture for hypergraphs can be answered: it was proved in Király (2013) that in every k-coloring of the edges of the r-uniform complete hypergraph K^r ($r \geq 3$), the vertex set of K^r can be covered by at most $\lceil k/r \rceil$ sets, each forming a connected hypergraph in some color.

Here we investigate the analogue problem for complete r-uniform r-partite hypergraphs. An edge coloring of a hypergraph is called **spanning** if every vertex is incident to edges of every color used in the coloring. We propose the following analogue of Ryser's conjecture.

In every spanning $(r+t)$-coloring of the edges of a complete r-uniform r-partite hypergraph, the vertex set can be covered by at most $t+1$ sets, each forming a connected hypergraph in some color.

We show that the conjecture (if true) is best possible. Our main result is that the conjecture is true for $1 \leq t \leq r-1$. We also prove a slightly weaker result for $t \geq r$, namely that $t+2$ sets, each forming a connected hypergraph in some color, are enough to cover the vertex set.

To build a bridge between complete r-uniform and complete r-uniform r-partite hypergraphs, we introduce a new notion. A hypergraph is complete r-uniform (r, ℓ)-partite if it has all r-sets that intersect each partite class in at most ℓ vertices (where $1 \leq \ell \leq r$).

Extending our results achieved for $\ell = 1$, we prove that for any $r \geq 3$, $2 \leq \ell \leq r$, $k \geq 1 + r - \ell$, in every spanning k-coloring of the edges of a complete r-uniform (r, ℓ)-partite hypergraph, the vertex set can be covered by at most $1 + \lfloor \frac{k-r+\ell-1}{\ell} \rfloor$ sets, each forming a connected hypergraph in some color.

1. Introduction

For an edge-colored hypergraph H let H_i denote its subhypergraph consisting of edges colored by i. The connected components of H_i are called monochromatic components of color i, and a **monochromatic component** refers to a monochromatic

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component of color i for some i. Here connectivity is understood in its weakest sense, a hypergraph is connected if either it has only one vertex or any two distinct vertices can be connected by a sequence of edges each intersecting the next. Every hypergraph can be uniquely partitioned into connected components. Components with a single vertex are called *trivial*. The connected components of H_i will by denoted by $H_i^1, H_i^2, \ldots, H_i^q$ if H_i has q components.

Given an edge-colored hypergraph H, let $c(H)$ denote the minimum integer m such that $V = V(H)$, the vertex set of H, can be covered by m monochromatic components of H. An edge coloring of a hypergraph is called **spanning** if every vertex is incident to edges of every color used in the coloring. Note that in spanning colorings every monochromatic component is non-trivial. The importance of this definition is shown in Theorem 1.1.

A well-known special case of a conjecture attributed to Ryser which actually appeared in [7] is that k-partite intersecting hypergraphs (hypergraphs in which any pair of edges have nonempty intersection) have transversals of at most $k-1$ vertices. An equivalent form is formulated in [5] as follows: if K is a complete graph with a k coloring on its edges, then $c(K) \le k-1$. The conjecture is true for $k \le 5$ and seems very difficult in general (further information can be found in [2,6]). A particular feature of the conjecture is that $c(K) \le k$ is obvious since the monochromatic stars centered at any vertex form monochromatic components. Note that the conjecture is obvious for colorings that are not spanning: if a vertex v is not incident to any edge in a specific color then the at most $k-1$ monochromatic stars centered at v form the required covering.

Surprisingly, the problem for hypergraphs is easier, Z. Király in [8] showed that if the edges of the complete r-uniform hypergraph K ($r \ge 3$) are colored with k colors, then $c(K) \le \lceil k/r \rceil$ and this is best possible (the $k = r$ case were already in [5] extending the well-known remark of Erdős and Rado stating that a graph or its complement is connected).

The problem naturally extends for sparser host graphs (or hypergraphs). Gyárfás and Lehel conjectured that for k-colored complete bipartite graphs G, $c(G) \le 2k-2$ (see [1]), here again $c(G) \le 2k-1$ is obvious. For the hypergraph case [3,4] initiated the study of $c(H)$ when H has bounded independence number.

The main subject of the present paper is the case when the target hypergraph K is a complete r-uniform r-partite hypergraph, i.e., when $V = V(K)$ is partitioned into nonempty classes $V_1 \cup \cdots \cup V_r$ and the edges of K are the sets containing one vertex from each class. Let $\mathrm{cov}(r, k)$ denote the maximum of $c(K)$ when K ranges over spanning k-colorings of complete r-uniform r-partite hypergraphs, and $\mathrm{COV}(r, k)$ denote the maximum of $c(K)$ when K ranges over (not necessarily spanning) k-colorings of complete r-uniform r-partite hypergraphs.

Throughout the paper we **always assume** $r \ge 3$. Our introductory theorem shows that only the spanning colorings are the interesting ones. For any positive integer k we use the standard notation $[k] = \{1, 2, \ldots, k\}$.

Theorem 1.1. *If* $r \ge 3$, *then* $\mathrm{COV}(r, k) = k$.

Proof. Let K be a k-edge-colored r-uniform r-partite complete hypergraph. Take an edge e of K. Let $C_1, \ldots, C_\ell$ be the monochromatic components with $|C_i \cap e| \ge r-1$. As $r > 2$, clearly no two of them have the same color, so $\ell \le k$. For every vertex $v \in V$ there is an edge $f \ni v$ with $|f \cap e| = r-1$, so v is covered by one of these components.

For the sharpness let $V_1 = [k]$ and color each edge e by color $e \cap V_1$. □

Remark 1.2. Since spanning colorings cannot have trivial components in any color, all monochromatic components meet every class in spanning colorings of r-uniform r-partite complete hypergraphs.

An edge of color i in a k-colored r-uniform hypergraph K is called **essential** if it is not contained in monochromatic components of any color different from i. When $\mathrm{cov}(r, k)$ is studied we may restrict ourselves to colorings having at least one essential edge in every used color, since otherwise a color can be eliminated by recoloring all edges of that color to some other color and the resulting hypergraph would still have a spanning coloring and the same set of (maximal) monochromatic components. This concept is established in [8] and works well in the proof of our initial result.

Theorem 1.3. $\mathrm{cov}(r, k) = 1$ *for every* $r \ge 3$ *and every* $1 \le k \le r$.

Proof. Let $e = \{v_1, \ldots, v_r\}$ be an essential edge of color 1 in a complete r-uniform r-partite hypergraph with vertex set $V = \cup_{i=1}^r V_i$ where $v_i \in V_i$. Let $R_i = e - \{v_i\}$ and denote by $\mathrm{Col}(R_i) \subseteq [k]$ the set of colors appearing on any edge of the form $R_i \cup \{v_i'\}$ (where $v_i' \in V_i$). Observe that $M = \mathrm{Col}(R_i) \cap \mathrm{Col}(R_j) = \{1\}$ for $i \ne j$. Indeed, $1 \in M$ because e is of color 1 and if $1 \ne c \in M$, then e is contained in the union of two intersecting edges of color c, contradicting the fact that e is essential (here we used $R_i \cap R_j \ne \emptyset$ because $r \ge 3$). By the pigeonhole principle there exists j such that $\mathrm{Col}(R_j) = \{1\}$. Now V_j is covered by the monochromatic component containing e (of color 1), and, as the coloring is spanning, it necessarily covers the whole V. □

By Theorem 1.3 from this point we may assume that $k = r + t$ with some integer $t \ge 1$.

Conjecture 1. $\mathrm{cov}(r, r+t) = t+1$ *for every* $r \ge 3,\ t \ge 1$.

It is worth formulating this conjecture in dual form. Assume K is a complete r-uniform r-partite hypergraph with a spanning k-coloring. Consider a new hypergraph H with vertex set $V(K)$ whose edges are the vertex sets of the monochromatic components in the coloring. The dual F (obtained by interchanging vertices and edges and keeping incidences) of this