



Extremal functions of forbidden multidimensional matrices

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ABSTRACT

Pattern avoidance is a central topic in graph theory and combinatorics. Pattern avoidance in matrices has applications in computer science and engineering, such as robot motion planning and VLSI circuit design. A d -dimensional zero–one matrix A avoids another d -dimensional zero–one matrix P if no submatrix of A can be transformed to P by changing some ones to zeros. A fundamental problem is to study the maximum number of nonzero entries in a d -dimensional $n \times \cdots \times n$ matrix that avoids P . This maximum number, denoted by $f(n, P, d)$, is called the extremal function.

We advance the extremal theory of matrices in two directions. The methods that we use come from combinatorics, probability, and analysis. Firstly, we obtain non-trivial lower and upper bounds on $f(n, P, d)$ when n is large for every d -dimensional block permutation matrix P . We establish the tight bound $\Theta(n^{d-1})$ on $f(n, P, d)$ for every d -dimensional tuple permutation matrix P . This tight bound has the lowest possible order that an extremal function of a nontrivial matrix can ever achieve. Secondly, we show that the limit inferior of the sequence $\{ \frac{f(n, P, d)}{n^{d-1}} \}$ has a lower bound $2^{\Omega(k^{1/2})}$ for a family of $k \times \cdots \times k$ permutation matrices P . We also improve the upper bound on the limit superior from $2^{O(k \log k)}$ to $2^{O(k)}$ for all $k \times \cdots \times k$ permutation matrices and show that the new upper bound also holds for tuple permutation matrices.

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1. Introduction

Pattern avoidance is a central topic in graph theory and combinatorics [2,7,14,16,18,21,24–26]. In this paper, we study d -dimensional matrices (or arrays) with only two distinct entries, 0 and 1, that avoid certain patterns. We are interested in the extremal problem of finding the largest number of ones in these matrices.

The extremal theory of matrices was initiated in papers [1,9,22] around 1990 to study specific problems arising in computational and discrete geometry. Mitchell produced an algorithm for computing a shortest rectilinear path avoiding rectilinear obstacles in the plane [22]. He showed that the complexity of this algorithm is bounded above by the solution of the extremal problem of certain matrices. Bienstock and Györi [1] found an upper bound on the solution of the extremal problem, thus bounding the complexity of the algorithm. Mitchell's algorithm can be directly applied to motion planning in robotics and wire routing in VLSI circuit design [20]. Füredi [9] used the extremal problem to derive an upper bound on Erdős–Moser [4] problem of determining the maximum number of unit distances in a convex polygon.

Matrix extremal problems also find applications to graph theory and other areas of combinatorics. If we associate 2-dimensional 0–1 matrices with ordered bipartite graphs by relating rows and columns to the two ordered partite sets of vertices and interpreting ones as edges, then this extremal problem can be viewed as the Turán extremal problem for ordered

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bipartite graphs [23]. The most recent surge in interest in the extremal theory of matrices is due to the resolution of the Stanley–Wilf conjecture in enumerative combinatorics using the extremal problem of matrices [17,21].

We denote a d -dimensional $n_1 \times \cdots \times n_d$ matrix by $A = (a_{i_1, \dots, i_d})$, where $1 \leq i_\ell \leq n_\ell$ for $\ell = 1, 2, \dots, d$. Matrix A is called a 0-1 matrix if all its entries are either 0 or 1. A multidimensional matrix is also called a multidimensional array in computer science. We may view a d -dimensional 0-1 matrix $A = (a_{i_1, \dots, i_d})$ geometrically as a d -dimensional rectangular box of lattice points with coordinates $(i_1, \dots, i_d)$. An ℓ -cross section of matrix A is the set of all the entries $a_{i_1, \dots, i_d}$ whose ℓ^{th} coordinates have the same value. An ℓ -row of matrix A is the collection of all the entries $a_{i_1, \dots, i_d}$ whose coordinates other than the ℓ^{th} coordinate have fixed values.

A d -dimensional $k \times \cdots \times k$ zero-one matrix is a permutation matrix if each of its ℓ -cross sections contains exactly one nonzero entry for every $\ell = 1, \dots, d$. The Kronecker product of two d -dimensional 0-1 matrices M and N , denoted by $M \otimes N$, is a d -dimensional matrix obtained by replacing each 1-entry of M with a copy of N and each 0-entry of M with a zero matrix the same size as N .

We say that a d -dimensional 0-1 matrix A contains another 0-1 matrix P if A has a submatrix that can be transformed into P by changing any number of ones to zeros. Otherwise, A is said to avoid P . Denote by $f(n, P, d)$ the maximum number of ones in a d -dimensional $n \times \cdots \times n$ zero-one matrix that avoids a given d -dimensional 0-1 matrix P . We are interested in the asymptotic behavior of the extremal function $f(n, P, d)$ for large n .

A motivation to study the extremal problem of multidimensional matrices comes from graph theory. A 2-dimensional $n \times n$ zero-one matrix can be represented as an ordered bipartite graph with partite sets of size n . Similarly, a d -dimensional 0-1 matrix corresponds to an ordered d -partite, d -uniform hypergraph. The extremal function $f(n, P, d)$ is the maximum number of hyperedges in an ordered d -partite, d -uniform hypergraph where each partite set has n vertices. Avoiding a given pattern P amounts to avoiding the corresponding ordered hypergraph.

It is easy to obtain trivial lower and upper bounds on $f(n, P, d)$.

Proposition 1.1. *If P is a 0-1 matrix that contains at least two ones, then $n^{d-1} \leq f(n, P, d) \leq n^d$.*

Proof. We can always choose a d -dimensional $n \times \cdots \times n$ zero-one matrix A , with 1-entries on a single ℓ -cross section for some ℓ and 0-entries elsewhere, such that A avoids P . Matrix A has exactly n^{d-1} ones, so the left inequality follows.

The right inequality follows from the fact that every d -dimensional $n \times \cdots \times n$ zero-one matrix has a total of n^d entries and that the matrix hence has at most n^d ones. $\square$

The upper bound in Proposition 1.1 is one order higher than the lower bound. The main problem is to improve the lower and upper bounds on $f(n, P, d)$ so that their orders are as close as possible.

The two-dimensional case of $d = 2$ has been well studied. Füredi and Hajnal conjectured that $f(n, P, 2) = O(n)$ for all permutation matrices P [10]. Klazar showed that this conjecture implies the Stanley–Wilf conjecture [17]. Marcus and Tardos proved the Füredi and Hajnal conjecture [21] and hence settled the Stanley–Wilf conjecture. Keszegh conjectured that $f(n, P, 2) = O(n)$ for all tuple permutation matrices P [15]. Geneson proved that the conjecture is true [11].

When R is a $k_1 \times k_2$ matrix of all ones, the extremal problem for $f(n, R, 2)$ is the matrix version of the classical Zarankiewicz problem. Kővári, Sós, and Turán found an upper bound $O(n^{2 - \frac{\max(k_1, k_2)}{k_1 k_2}})$ on $f(n, R, 2)$ [19]. A lower bound $\Omega(n^{2 - \frac{k_1 + k_2 - 2}{k_1 k_2 - 1}})$ was also known [5]. Hesterberg extended these bounds to $f(n, P \otimes R, 2)$ where P is a permutation matrix [12,13].

Pach and Tardos showed that $f(n, P, 2)$ is super-additive in n [23]. When P is a permutation matrix, the sequence $\{ \frac{f(n, P, 2)}{n} \}$ is bounded because of $f(n, P, 2) = O(n)$ and is therefore convergent by Fekete's lemma on super-additive sequences [6]. The limit is known as the Füredi–Hajnal limit and has recently attracted great attention [3,7,8]. Cibulka [3] showed that this limit is always at least $2(k-1)$ when P is a $k \times k$ permutation matrix and that the limit is exactly $2(k-1)$ when P is the identity matrix. Fox showed that the Füredi–Hajnal limit has a lower bound $2^{\Omega(k^{\frac{1}{2}})}$ for a family of $k \times k$ permutation matrices [8]. Marcus and Tardos [21] showed that this limit has an upper bound $2^{O(k \log k)}$ for every $k \times k$ permutation matrix P , and Fox [7] improved this upper bound to $2^{O(k)}$.

Little has been done on the multidimensional case. Klazar and Marcus [18] studied the extremal function when the d -dimensional matrix P is a permutation matrix of size $k \times \cdots \times k$ and found $f(n, P, d) = O(n^{d-1})$, generalizing the $d = 2$ results [21]. In particular, they showed that $\frac{f(n, P, d)}{n^{d-1}} = 2^{O(k \log k)}$, which is the multidimensional generalization of the Marcus and Tardos upper bound on the Füredi–Hajnal limit [21].

In this paper, we advance the extremal theory of matrices in two directions. In the first direction, we study the extremal functions $f(n, R, d)$ and $f(n, P \otimes R, d)$, where P is a permutation matrix and R is a $k_1 \times \cdots \times k_d$ matrix of ones only; matrix $P \otimes R$ is called a block permutation matrix. We show that both $f(n, R, d)$ and $f(n, P \otimes R, d)$ have a lower bound $\Omega(n^{d-\beta})$ and an upper bound $O(n^{d-\alpha})$, where $\alpha = \frac{\max(k_1, \dots, k_d)}{k_1 \cdot k_2 \cdots k_d}$ and $\beta = \frac{k_1 + \cdots + k_d - d}{k_1 \cdot k_2 \cdots k_d - 1}$. These bounds significantly improve the trivial ones given in Proposition 1.1.

Both the lower bound $\Omega(n^{d-\beta})$ and the upper bound $O(n^{d-\alpha})$ are in the same order as $\Theta(n^{d-1})$ whenever $\alpha = \beta$. This is exactly when only one of $k_1, \dots, k_d$ of R is larger than 1; the corresponding $P \otimes R$ is called a tuple permutation matrix.

When $\alpha \neq \beta$, our results on $f(n, R, d)$ generalize the Kővári–Sós–Turán upper bound [19] from two dimensions to higher dimensions. Our results on $f(n, P \otimes R, d)$ also extend Hesterberg's results [12] from $d = 2$ to $d > 2$.

When $\alpha = \beta$, our result that $f(n, P, d) = \Theta(n^{d-1})$ for every d -dimensional tuple permutation matrix P , on one hand, generalizes Geneson's result [11] from $d = 2$ to $d \geq 2$. On the other hand, even when $d = 2$ our ideas improve some key

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