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Forbidden set of induced subgraphs for 2-connected supereulerian graphs

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ABSTRACT

Let $\mathcal{H} = \{H_1, \dots, H_k\}$ be a set of connected graphs. A graph is said to be $\mathcal{H}$-free if it does not contain any member of $\mathcal{H}$ as an induced subgraph. We show that if the following statements hold,

- $|\mathcal{H}| \geq 2$,
- $K_{1,3} \in \mathcal{H}$,
- for any integer n_0, every 2-connected $\mathcal{H}$-free graph G of order at least n_0 is supereulerian, *i.e.*, G has a spanning closed trail,

then $\mathcal{H} \setminus \{K_{1,3}\}$ contains an $N_{i,j,k}$ or a path where $N_{i,j,k}$ denotes the graph obtained by attaching three vertex-disjoint paths of lengths $i, j, k \geq 0$ to a triangle.

As an application, we characterize all the forbidden triples $\mathcal{H}$ with $K_{1,3} \in \mathcal{H}$ such that every 2-connected $\mathcal{H}$-free graph is supereulerian. As a byproduct, we also characterize minimal 2-connected non-supereulerian claw-free graphs.

1. Introduction

For graph-theoretic notation not explained in this paper, we refer the reader to [2]. In this paper, we consider only simple graphs except the unique graph C_2 with two vertices and two edges. Let G be a graph, we use $\langle F\rangle_G$ to denote the *induced subgraph* of G by the subset F of $V(G)$. The set of *neighbors* of a vertex v in a graph G is denoted by $N_G(v)$. We denote by $P(u, v)$ a path between two vertices u and v.

Following [3], we denote by $\mathcal{P}$ the class of all graphs that are obtained by taking two vertex-disjoint triangles $a_1a_2a_3a_1$, $b_1b_2b_3b_1$ and by joining every pair of vertices $\{a_i, b_i\}$ by a copy of a path $P_{k_i} = a_ic_i^1c_i^2\cdots c_i^{k_i-2}b_i$ with $k_i \geq 3$ or by a triangle $a_ib_ic_ia_i$. We denote graphs from $\mathcal{P}$ by P_{x_1,x_2,x_3}, where $x_i = k_i$ if a_i, b_i are joined by a copy of P_{k_i}, and $x_i = T$ if a_i, b_i are joined by a copy of $T = a_ib_ic_ia_i$. We define $\mathcal{P}^* = \{P_{x_1,x_2,x_3} \in \mathcal{P} : x_1, x_2, x_3 \neq T \text{ and } x_3 \geq x_2 \geq x_1 \geq 3\}$.

A graph is called *Hamiltonian* if it contains a Hamilton cycle, i.e., a cycle containing all of its vertices. A subgraph H of a graph G is even if every vertex of H has even degree. A graph is called *supereulerian* if it has a spanning even connected subgraph. A *pendant* vertex is a vertex of degree 1. We use $P_i (i \geq 1)$ to denote the path on i vertices and $C_i (i \geq 2)$ the cycle on i vertices. We use $N_{i,j,k}$ to denote the *generalized* net obtained by attaching three vertex-disjoint paths of lengths $i, j, k \geq 0$ to a triangle, respectively.

Let $\mathcal{H} = \{H_1, \dots, H_k\}$ be a set of connected graphs. A graph G is said to be $\mathcal{H}$-free if G does not contain H as an induced subgraph for all H in $\mathcal{H}$, and we call each graph H of $\mathcal{H}$ a forbidden subgraph. Remark to $\mathcal{H}$-free property, we assume that for

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each pair of R, S of $\mathcal{H}$, R is not an induced subgraph S and S is also not an induced subgraph of R. When we consider $\mathcal{H}$-free property such that $|\mathcal{H}| \geq 2$, each member of $\mathcal{H}$ is not P_2, P_3 because P_2-free and P_3-free connected graphs are empty graphs and complete graphs, respectively. We call $\mathcal{H}$ a forbidden pair if $|\mathcal{H}| = 2$, and we call $\mathcal{H}$ a forbidden triple if $|\mathcal{H}| = 3$. For convenience, we sometimes abbreviate the term the set of connected graphs $H_1, \ldots, H_k$ to $H_1 \cdots H_k$.

In order to state results on forbidden subgraphs clearly, we follow the following notation introduced in [5] and [11]. For two sets $\mathcal{H}_1$ and $\mathcal{H}_2$ of connected graphs, we write $\mathcal{H}_1 \preceq \mathcal{H}_2$ if for every graph H_2 in $\mathcal{H}_2$, there exists a graph H_1 in $\mathcal{H}_1$ such that H_1 is an induced subgraph of H_2. By the definition of the relation "$\preceq$", if $\mathcal{H}_1 \preceq \mathcal{H}_2$, then $\mathcal{H}_1$-free graph is also $\mathcal{H}_2$-free.

For a property $\mathcal{Q}$, it is a popular research topic to give forbidden subgraphs conditions forcing a graph to have the property $\mathcal{Q}$, see, e.g. for Hamiltonian properties in [7], [8–10], [12,13]; for the existence of perfect matchings in [11,16]; for the existence of dominating cycles in [5].

We say that a graph G is *minimal* with respect to a property $\mathcal{Q}$ if there exists no proper induced subgraph of G with property $\mathcal{Q}$. We start with the result of the characterization of forbidden pairs for 2-connected supereulerian graphs.

Theorem 1 (*Lv and Xiong [15]*). *Let $\mathcal{H}$ be a forbidden pair. Then every 2-connected $\mathcal{H}$-free graph of order at least 6 is supereulerian if and only if $\mathcal{H} \preceq \{K_{1,3}, P_7\}$, $\mathcal{H} \preceq \{K_{1,3}, N_{0,1,3}\}$, $\mathcal{H} \preceq \{K_{1,3}, N_{1,1,2}\}$, or $\mathcal{H} \preceq \{K_{1,4}, P_5\}$.*

Liu Xia [17] pointed out that there would be an additional forbidden pair $\{C_4, P_5\}$ in Theorem 1. Bedrossian characterized forbidden pairs for Hamiltonicity of 2-connected graphs.

Theorem 2 (*Bedrossian [1]*). *Let $\mathcal{H}$ be a forbidden pair. Then every 2-connected $\mathcal{H}$-free graph is Hamiltonian if and only if $\mathcal{H} \preceq \{K_{1,3}, P_6\}$, $\mathcal{H} \preceq \{K_{1,3}, N_{0,1,2}\}$, or $\mathcal{H} \preceq \{K_{1,3}, N_{1,1,1}\}$.*

In [7], the forbidden pairs for Hamiltonicity of 2-connected graphs have been completely determined when we allow a finite number of exceptions.

Theorem 3 (*Faudree and Gould [7]*). *Let $\mathcal{H}$ be a forbidden pair. Then every 2-connected $\mathcal{H}$-free graph of sufficiently large order is Hamiltonian if and only if $\mathcal{H} \preceq \{K_{1,3}, P_6\}$, $\mathcal{H} \preceq \{K_{1,3}, N_{0,0,3}\}$, $\mathcal{H} \preceq \{K_{1,3}, N_{0,1,2}\}$, or $\mathcal{H} \preceq \{K_{1,3}, N_{1,1,1}\}$.*

Following the results above, if we plus one forbidden connected graph X to force a 2-connected claw-free graph to be supereulerian (Hamiltonian, respectively), then X must be an induced subgraph of $N_{i,j,k}$ or P_l. It is natural to ask the problem: If we plus $k(k \geq 2)$ forbidden connected graphs $X_1, \ldots, X_k$ to force a 2-connected claw-free graph to be supereulerian (Hamiltonian, respectively), whether one of these X_i is still an $N_{i,j,k}$ or a path? In this paper, we shall prove the following result, which give a positive answer for this problem.

Theorem 4. *Let $\mathcal{H} = \{K_{1,3}, H_1, \ldots, H_k\}$ be a set of connected graphs. For any integer n_0, every 2-connected $\mathcal{H}$-free graph with order at least n_0 being supereulerian (Hamiltonian, respectively) implies that one of $\{H_1, \ldots, H_k\}$ is an $N_{i,j,k}$ or a path.*

Brousek later characterized forbidden triples $\mathcal{H}$ with $K_{1,3} \in \mathcal{H}$ such that every 2-connected $\mathcal{H}$-free graph is Hamiltonian.

Theorem 5 (*Brousek [4]*). *Let $\mathcal{H}$ be a forbidden triple such that $K_{1,3} \in \mathcal{H}$ and each member of $\mathcal{H}$ is not isomorphic to an induced subgraph of $P_6, N_{0,1,2}$, or $N_{1,1,1}$, and let G be a 2-connected $\mathcal{H}$-free graph. Then G is Hamiltonian if and only if $\mathcal{H} \preceq \{K_{1,3}, P_7, P_{T,T,T}\}$, $\mathcal{H} \preceq \{K_{1,3}, N_{1,1,2}, P_{T,T,T}\}$, $\mathcal{H} \preceq \{K_{1,3}, N_{0,1,2}, P_{3,3,3}\}$, or $\mathcal{H} \preceq \{K_{1,3}, N_{0,1,3}, R\}$, where R is the graph obtained by removing one vertex of degree four from a copy of $P_{T,T,T}$.*

We also apply Theorem 4 to characterize all the forbidden triples $\mathcal{H}$ with $K_{1,3} \in \mathcal{H}$ such that every 2-connected $\mathcal{H}$-free graph is supereulerian. Remark that with respect to Theorem 1, we will consider that each graph of $\mathcal{H}$ is not isomorphic to an induced subgraph of P_7, $N_{0,1,3}$ or $N_{1,1,2}$.

Before stating the following result we need to define three families of forbidden triples. Set $\mathbf{H}_1 = \{\{K_{1,3}P_{3,3,3}X\} : X \in \{P_8, N_{0,1,4}, N_{1,1,3}, N_{1,2,2}\}\}$, $\mathbf{H}_2 = \{\{K_{1,3}XY\} : X \in \{B_1, B_2\}, Y \in \{P_9, N_{0,1,5}, N_{1,1,4}, N_{1,2,3}, N_{2,2,2}\}\}$, $\mathbf{H}_3 = \{\{K_{1,3}B_3X\} : X \in \{P_{10}, N_{2,2,3}, N_{0,2,5}, N_{1,2,4}\}\}$, where $P_{3,3,3}, B_1, B_2, B_3$ are depicted in Fig. 1.

Theorem 6. *Let $\mathcal{H}$ be a forbidden triple such that $K_{1,3} \in \mathcal{H}$ and each member of $\mathcal{H}$ is not isomorphic to an induced subgraph of $P_7, N_{0,1,3}$ or $N_{1,1,2}$. Then G being a 2-connected $\mathcal{H}$-free graph implies that G is supereulerian if and only if $\mathcal{H} \preceq \mathcal{H}_i$ for some $\mathcal{H}_i \in \mathbf{H}_1 \cup \mathbf{H}_2 \cup \mathbf{H}_3$.*

Following the proof of necessity of Theorem 6 in Section 3, we shall see that many such triples were excluded because of some graphs with small order. So it is naturally to ask the following question.

Question 1. What are the forbidden triples XYZ with $X = K_{1,3}$ such that every 2-connected XYZ-free graph of sufficiently large order is supereulerian; but no pair of the triple XYZ has this property?

Faudree et al. [9] considered the similar question of which triples (including $K_{1,3}$) of forbidden subgraphs potentially imply all sufficiently large graphs are Hamiltonian, but they did not determine all forbidden triple $\mathcal{H}$ with $K_{1,3} \in \mathcal{H}$ such that every 2-connected $\mathcal{H}$-free graph is Hamiltonian and it is still open for those graphs with sufficient large order. So we think Question 1 is also difficult, we can consider those forbidden triples (including $K_{1,3}$ and some induced subgraph of $P_{3,3,3}$) of forbidden subgraphs potentially imply all sufficiently large graphs are supereulerian and we can obtain the following.