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Investigations on association schemes with elementary abelian thin residue

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ABSTRACT

In this paper, we give a new class of association schemes whose thin residues are isomorphic to an elementary abelian p-group of order p^2. We then study the automorphism groups of these schemes and determine whether these schemes are schurian.

1. Introduction

Association schemes in which the thin residues are thin have been studied in several papers; for example, see [1,2,5,6]. The study and classification of association schemes whose thin residues are elementary abelian p-groups has been pursued by the first author with Cho and Hirasaka over the years. Their main findings may be summarized as follows. Let p be prime, and let $\mathcal{C}_p$ denote the collection of association schemes each of which has the properties that (i) every valency of the scheme is either 1 or p; (ii) the thin radical of the scheme is isomorphic to an elementary abelian p-group of order p^2; and (iii) the thin residue and thin radical of the scheme are the same. Cho, Hirasaka and Kim [1] have shown that the index of the thin residue of every scheme in $\mathcal{C}_p$ does not exceed p^2+p+1. Hirasaka and Kim [2] have shown that the index of thin residue of each scheme in $\mathcal{C}_p$ is at least 3, and that there exists a scheme in $\mathcal{C}_p$ whose index of thin residue is δ for each δ with $3 \le \delta \le p+2$, $\delta = p^2$ and $\delta = p^2+p+1$. They have also shown that there exists a nonschurian scheme in $\mathcal{C}_p$ whose index of the thin residue is δ for each δ with $4 \le \delta \le p+2$.

In this paper, for any prime number q and positive integer $m > 1$ such that $\frac{q^m-1}{q-1} \le p+1$, we give a construction of association schemes belonging to $\mathcal{C}_p$ and whose index of thin residue is q^m. (So the index lies between $p+2$ and p^2 for many prime numbers q and positive integers $m > 1$.) We then study the automorphism groups of these association schemes and determine which of these schemes are schurian. In order to describe these works more precisely, we shall first fix some notation and briefly connect the notion of an association scheme to related concepts and topics in group theory.

Let X be a finite nonempty set. We use 1_X to denote the set of all pairs (x, x) with $x \in X$. For each subset s of the cartesian product $X \times X$, we define s^* to be the set of all pairs (y, z) with $(z, y) \in s$. For x an element of X and r a subset of $X \times X$, we denote by xr the set of all elements y in X with $(x, y) \in r$. Now we fix a partition S of $X \times X$ with $\emptyset \notin S$ and $1_X \in S$, and we

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assume that $s^* \in S$ for each element s in S. The set S is called an *association scheme on* X if, for any three elements p, q and r in S, there exists a cardinal number c_{pq}^r such that $|yp \cap zq^*| = c_{pq}^r$ for any two elements y, z in X with (y, z) in r.

In terms of Zieschang's [3] description, the notion of an association scheme generalizes in a natural way the notion of a group in the following sense. Let S be an association scheme on X, and let P and Q be nonempty subsets of S. We define PQ to be the set of all elements s in S for which there exist elements p in P and q in Q satisfying $c_{pq}^s \geq 1$. If P or Q is a singleton set with $\{p\} = P$ or $\{q\} = Q$ we write pQ or Pq (or pq) instead of PQ. The set PQ is called the *complex product* of P and Q. The associated operation on the set of all nonempty subsets of S is referred to as the *'complex multiplication'* in S. It follows that, for any two elements p and q of an association scheme S, p^*q is not empty. A subset T of S is called *closed* if $\bigcup_{t\in T} t$ is an equivalence relation on X. For a closed subset T of S, we define the set of *cosets* of T in X by $X/T := \{xT \mid x \in X\}$ where $xT := \bigcup_{t\in T} xt$. We also define $S/\!\!/T := \{s^T \mid s \in S\}$ where $s^T := \{(xT, yT) \mid y \in xTsT\}$. Then $S/\!\!/T$ is an association scheme on X/T called as the *factor* scheme of S over T (see [4, Theorem 4.1.3]).

An element s of an association scheme S on X is called *thin* if s^*s contains exactly one element which then must be the identity element 1_X of S. The *valency* of s, denoted by n_s, is the integer $c_{ss^*}^{1_X}$, which is the cardinality $|xs|$ for any $x \in X$. In terms of valency, s is thin if and only if $n_s = 1$. A nonempty subset of S is called *thin* if each element of the subset is thin.

Given a thin association scheme S, the set $S^\wedge$ of all sets $\{s\}$ with $s \in S$ is a group with respect to the restriction of the complex multiplication in S to $S^\wedge$. Conversely, let G be a group. For each element g in G, we define $g^\vee$ to be the set of all pairs (a, b) of elements of G satisfying $ag = b$. Then the set $G^\vee$ of all sets $g^\vee$ with $g \in G$ is a thin association scheme on G. It turns out that, for each thin association scheme S, the association scheme $S^{\wedge\vee}$ is isomorphic to S. Conversely, for each group G, the group $G^{\vee\wedge}$ is isomorphic to G. This one-to-one correspondence between groups and thin association schemes has a natural generalization (see [3, Appendix Theorem A]).

Let G be a group, and let H be a subgroup of G. For each element g in G, we define g^H to be the set of all pairs (aH, bH) satisfying $a \in G$ and $b \in aHgH$. Setting

$$C[G, H] := \{gH \mid g \in G\}$$

and

$$G/\!\!/H := \{g^H \mid g \in G\},$$

it is clear that $G/\!\!/H$ is an association scheme on $C[G, H]$.

An association scheme S is called *schurian* if there exist groups G and H such that H is a subgroup of G and S is isomorphic to $G/\!\!/H$. We note that the case $H = \{1\}$ just says that thin association schemes are schurian. Given an association scheme S, the closed subset $\{s \in S \mid n_s = 1\}$ is called the *thin radical* of S and denoted by $\mathbf{O}_\theta(S)$. A closed subset T of S is called *strongly normal* in S, denoted by $T \lhd^\sharp S$, if $s^*Ts \subseteq T$ for every $s \in S$. We denote the smallest closed subset $\bigcap_{T\lhd^\sharp S} T$ of S by $\mathbf{O}^\theta(S)$, and call it the *thin residue* of S. It follows from [3, Theorem 2.3.1] that $\mathbf{O}^\theta(S)$ is strongly normal in S, and

$$\mathbf{O}^\theta(S) = \left\langle \bigcup_{s\in S} s^*s \right\rangle. \tag{1}$$

Furthermore, $S/\!\!/\mathbf{O}^\theta(S)$ is thin (see [3, Theorem 2.3.4]). For a closed subset T of S, n_S/n_T is called the *index* of T in S.

Now given a prime integer p, we use C_p to denote the cyclic group of order p. Let S be an association scheme satisfying the following two conditions:

$$\mathbf{O}^\theta(S) \text{ is a thin closed subset of } S \text{ isomorphic to } C_p \times C_p, \text{ and} \tag{2}$$

$$\{n_s \mid s \in S \setminus \mathbf{O}^\theta(S)\} = \{p\}. \tag{3}$$

Then, using δ to denote the index $n_S/n_{\mathbf{O}^\theta(S)}$, it was shown that

$$3 \leq \delta \leq p^2 + p + 1$$

in [1, Corollary 2.8] and [2, Proposition 3.1]. For the cases when $3 \leq \delta \leq p + 2$, $\delta = p^2$ and $\delta = p^2 + p + 1$, the association schemes satisfying (2) and (3) have been constructed and studied in [2]. We shall construct and study the association schemes whose δ satisfies $p + 2 \leq \delta \leq p^2$ in the current paper.

In order to describe our results, we recall the definition of the automorphism group of an association scheme. Let R and S be association schemes on W and X, respectively. A bijection ϕ from $W \cup R$ to $X \cup S$ is called an *isomorphism* if it satisfies the following conditions:

(i) $\phi(W) \subseteq X$ and $\phi(R) \subseteq S$;
(ii) for all $v, w \in W$ and $r \in R$ with $(v, w) \in r$, $(\phi(v), \phi(w)) \in \phi(r)$.

An *automorphism* of a scheme S on X is an isomorphism ϕ from $X \cup S$ to itself such that $\phi(s) = s$ for all $s \in S$. We denote the set of automorphisms of S by $\mathrm{Aut}(S)$. Note that $\mathrm{Aut}(S)$ is a group, called the *automorphism group* of S.