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Counting numerical semigroups by genus and even gaps

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ABSTRACT

Let n_g be the number of numerical semigroups of genus g. We present an approach to compute n_g by using even gaps, and the question: Is it true that $n_{g+1} > n_g$? is investigated. Let $N_\gamma(g)$ be the number of numerical semigroups of genus g whose number of even gaps equals γ. We show that $N_\gamma(g) = N_\gamma(3\gamma)$ for $\gamma \le \lfloor g/3 \rfloor$ and $N_\gamma(g) = 0$ for $\gamma > \lfloor 2g/3 \rfloor$; thus the question above is true provided that $N_\gamma(g+1) > N_\gamma(g)$ for $\gamma = \lfloor g/3 \rfloor + 1, \ldots, \lfloor 2g/3 \rfloor$. We also show that $N_\gamma(3\gamma)$ coincides with f_γ, the number introduced by Bras-Amorós (2012) in connection with semigroup-closed sets. Finally, the stronger possibility $f_\gamma \sim \varphi^{2\gamma}$ arises being $\varphi = (1+\sqrt{5})/2$ the golden number.

1. Introduction

A *numerical semigroup* S is a submonoid of the set of nonnegative integers $\mathbb{N}_0$, equipped with the usual addition, such that $G(S) := \mathbb{N}_0 \setminus S$, the set of *gaps* of S, is finite. The number of elements $g = g(S)$ of $G(S)$ is called the *genus* of S and thus the semigroup property implies (see e.g. [16, Lemma 2.14])

$$S \supseteq \{2g + i : i \in \mathbb{N}_0\}\,. \tag{1.1}$$

Suitable references for the background on numerical semigroups that we assume are in fact the books [16,27]. In spite of its simplicity, as a mathematical object, a numerical semigroup often plays a key role in the study of more involved or subtle structures arising e.g. in Algebraic Curve Theory [15,20,22,24,33] or in Coding Theory [6,26].

In this paper we deal with a problem of purely combinatorial nature, namely: for $g \in \mathbb{N}_0$ given, find the number n_g of elements of the family $\mathcal{S}_g$ of numerical semigroups of genus g; Kaplan [18] wrote a nice survey and state of the art on this problem, and one can find information on these numbers in Sloane's On-line Encyclopedia of Integer Sequences [31]. Indeed, our goal here is the question (1.2) below.

We have $n_g \le \binom{2g-1}{g}$ by (1.1) and in fact, a better bound is known, namely $n_g \le \frac{1}{g+1}\binom{2g}{g}$ which was obtained by Bras-Amorós and de Mier via so-called Dyck paths [8]. Further bounds on n_g were computed by Bras-Amorós [4] via the semigroup tree method; see also Bras-Amorós and Bulygin [7], O'Dorney [23], Elizalde [13]. Blanco and Rosales [2] approached this problem by considering a partition of $\mathcal{S}_g$ by subsets of semigroups S of a given Frobenius number $F = F(S)$, which by definition is the biggest integer which does not belong to S; see also [1]. In any case, computing the exact value of n_g seems to be out of reach although there exist algorithmic methods for determining such a number [9,14].

By taking into consideration the first 50 values of n_g, Bras-Amorós [3] conjectured Fibonacci-like properties on the behavior of the sequence n_g:

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(A) $n_{g+2} \geq n_{g+1} + n_g$ for any g;
(B) $\lim_{g\to\infty} \frac{n_{g+1}+n_g}{n_{g+2}} = 1$;
(C) $\lim_{g\to\infty} \frac{n_{g+1}}{n_g} = \varphi := \frac{1+\sqrt{5}}{2}$, so-called *golden number*.

Indeed, Conjectures (B) and (C) have been recently proved by Zhai [35]. Here we focus in the following problem suggested by (A) whose answer is positive for large g by (C) or $g \leq 50$ by the aforementioned values in [3] (which were recently extended to $g \leq 67$ in [14]):

$$\text{Is it true that } n_{g+1} > n_g \text{ for any } g \geq 1? \tag{1.2}$$

The *multiplicity* $m(S)$ of a numerical semigroup S is its first positive element. Kaplan [19] gave an approach to Conjecture A and Question (1.2) by counting numerical semigroups by genus and multiplicity. He obtained some partial interesting results, but his method does not solve the problems.

In addition, Bras-Amorós [5] introduced the notion of *ordinarization transform* $\mathbf{T} : \mathcal{S}_g \to \mathcal{S}_g$ given by $\mathbf{T}(S) = (S \cup \{F(S)\}) \setminus \{m(S)\}$, with $S \neq S_g := \{0\} \cup \{g+i : i \in \mathbb{N}\}$ (so-called *ordinary semigroup* of genus g). Then the minimum nonnegative integer r such that $\mathbf{T}^r(S) = S_g$ is the *ordinarization number* of S; it turns out that $r \leq g/2$, and so she counted numerical semigroups by genus and ordinarization number. Unfortunately this method also does not give an answer to either computing n_g or question (1.2).

In this paper we approach (1.2) by counting numerical semigroups by genus and number of even gaps. Our method is motivated by the interplay between double covering of curves and Weierstrass semigroups at totally ramified points of such coverings; see for instance Kato [20], Garcia [15], Torres [33], Oliveira and Pimentel [24], Komeda [22].

Let $N_\gamma(g)$ denote the number of elements of the family $\mathcal{S}_\gamma(g)$, so-called *γ-hyperelliptic semigroups of genus g* ; i.e. those in $\mathcal{S}_g$ whose number of even gaps equals γ. From Corollary 2.4

$$n_g = \sum_{\gamma=0}^{\lfloor 2g/3 \rfloor} N_\gamma(g)\,; \tag{1.3}$$

in particular, see Remark 3.5, Question (1.2) holds true provided that

$$N_\gamma(g+1) > N_\gamma(g) \quad \text{for } \gamma = \lfloor g/3 \rfloor + 1, \ldots, \lfloor 2g/3 \rfloor\,. \tag{1.4}$$

In Section 2 we deal with the set of even gaps of a numerical semigroup, where the key result is Lemma 2.3 (cf. [34]). In particular, (1.3) is a direct consequence of the stratification in (2.2). For $2g \geq 3\gamma$ (cf. [32]) we point out a quite useful parametrization, namely $\mathcal{S}_\gamma(g) \to \mathcal{S}_\gamma$, $S \mapsto S/2$, which was introduced by Rosales et al. [29] (see (2.3), [17,28]). Thus Remark 2.11 shows the class of numerical semigroups we deal with in this paper; we do observe that these semigroups were already studied for example in [25] by using the concept of *weight* of semigroups.

We have $N_\gamma(g) \leq N_\gamma(3\gamma)$ and $N_\gamma(g) = N_\gamma(3\gamma)$ if and only if $g \geq 3\gamma$; see Corollary 3.4. The key ingredient here is the t-translation of a numerical semigroup introduced in Definition 3.1.

By the above considerations on $N_\gamma(g)$, it is natural to investigate the asymptotic behavior of the sequence $N_\gamma(3\gamma)$ which is studied in Section 4; indeed, to our surprise, it coincides with the sequence f_γ, introduced by Bras-Amorós in [5, p. 2515], which has to do with semigroup-closed sets (see Theorem 4.4 here).

Finally in Section 5 we compute certain limits involving f_γ (see Proposition 5.1) which are of theoretical interest as they are related to the stronger possibility: $f_\gamma \sim \varphi^{2\gamma}$.

2. On the even gaps of a numerical semigroup

Throughout, let S be a numerical semigroup of genus $g = g(S)$, $G_2 = G_2(S)$ the set of its even gaps, and $\gamma = \gamma(S)$ the number of elements of G_2. As a matter of terminology, we say that S is *γ-hyperelliptic*. In particular, from (1.1), there are exactly $g - \gamma$ (resp. γ) even (resp. odd) nongaps in $S \cap [1, 2g]$. For $\gamma \geq 1$, these odd nongaps will be denoted by

$$o_\gamma = o_\gamma(S) < \cdots < o_1 = o_1(S)\,. \tag{2.1}$$

Remark 2.1. With notation as above, we notice that $o_i \leq 2g - 2i + 1$ for $i = 1, \ldots, \gamma$.

As usual, for pairwise different natural numbers $a_1, \ldots, a_\alpha$, we set $\langle a_1, \ldots, a_\alpha \rangle := \{a_1x_1 + \cdots + a_\alpha x_\alpha : x_1, \ldots, x_\alpha \in \mathbb{N}_0\}$. It is well-known, so far, that this set is a numerical semigroup if and only if $\gcd(a_1, \ldots, a_\alpha) = 1$.

Remark 2.2. We have $\gamma(S) = 0$ if and only $S = \langle 2, 2g+1 \rangle$; in the literature, this semigroup is classically called *hyperelliptic*. In general $g \geq \gamma$, and equality holds if and only if $g = \gamma = 0$.

From now on, we always assume $\gamma \geq 1$ so that $1, 2 \in G(S)$, the set of gaps of S, and $g \geq \gamma + 1$.

The following result and their corollaries were already noticed in [34]. It is analogous to (1.1), and for the sake of completeness we state proofs.