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Arc-disjoint hamiltonian paths in non-round decomposable local tournaments

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ABSTRACT

Thomassen proved that a strong tournament T has a pair of arc-disjoint Hamiltonian paths with distinct initial vertices and distinct terminal vertices if and only if T is not an almost transitive tournament of odd order, where an almost transitive tournament is obtained from a transitive tournament with acyclic ordering $u_1, u_2, \dots, u_n$ (i.e., $u_i \to u_j$ for all $1 \le i < j \le n$) by reversing the arc u_1u_n. A digraph D is a local tournament if for every vertex x of D, both the out-neighbors and the in-neighbors of x induce tournaments. Bang-Jensen, Guo, Gutin and Volkmann split local tournaments into three subclasses: the round decomposable; the non-round decomposable which are not tournaments; the non-round decomposable which are tournaments. In 2015, we proved that every 2-strong round decomposable local tournament has a Hamiltonian path and a Hamiltonian cycle which are arc-disjoint if and only if it is not the second power of an even cycle. In this paper, we discuss the arc-disjoint Hamiltonian paths in non-round decomposable local tournaments, and prove that every 2-strong non-round decomposable local tournament contains a pair of arc-disjoint Hamiltonian paths with distinct initial vertices and distinct terminal vertices. This result combining with the one on round decomposable local tournaments extends the above-mentioned result of Thomassen to 2-strong local tournaments.

1. Terminology and introduction

In this paper, we consider finite digraphs without loops and multiple arcs. The main source for terminology and notation is [3].

For an integer n, $[n]$ will denote the set $\{1, 2, 3, \dots, n\}$.

Let $D = (V, A)$ be a digraph, if there is an arc from a vertex x to y, we say that x dominates y and denote it by $x \to y$. If V_1 and V_2 are disjoint subsets of vertices of D such that there is no arc from V_2 to V_1 and $a \to b$ for all $a \in V_1$ and $b \in V_2$, then we say that V_1 completely dominates V_2 and denote this by $V_1 \Rightarrow V_2$. We shall use the same notation when V_1 and V_2 are subdigraphs of D. Let $N^-(x)$ (respectively, $N^+(x)$) denote the set of vertices dominating (respectively, dominated by) x in D and say that $N^-(x)$ (respectively, $N^+(x)$) is the in-neighborhood of x (respectively, the out-neighborhood of x). The vertices in $N^-(x)$ and $N^+(x)$ are called the in-neighbors and out-neighbors of x.

Let H be a subdigraph of D, if $V(D) = V(H)$, we say that H is a spanning subdigraph of D. If every arc of $A(D)$ with both end-vertices in $V(H)$ is in $A(H)$, we say that H is induced by $X = V(H)$ and denote this by $D\langle X\rangle$. We also use the notation $D - X$, where $X \subseteq V$, for the digraph $D\langle V(D) \setminus V(X)\rangle$.

Let D_1, D_2 be two subdigraphs of a digraph D. The union $D_1 \cup D_2$ is the digraph D with vertex set $V(D_1) \cup V(D_2)$ and arc set $A(D_1) \cup A(D_2)$.

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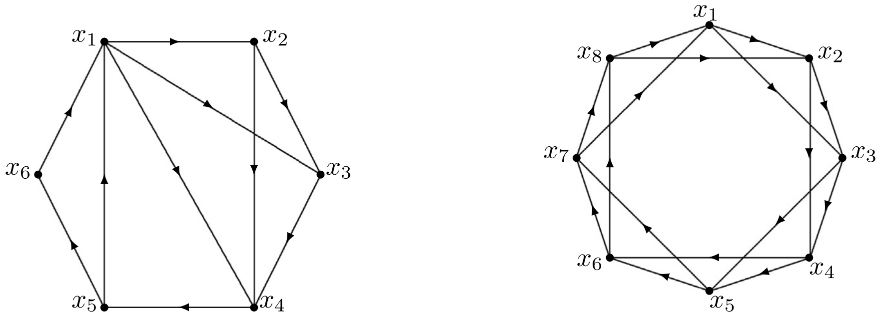

(a) R. (b) C_8^2.

Fig. 1. A round digraph and the second power of an 8-cycle.

Paths and cycles in a digraph are always directed. Let P be a directed path of a digraph D. If $V(P) = V(D)$, then P is a Hamiltonian path of D. Similarly, let C be a directed cycle of a digraph D. If $V(C) = V(D)$, then C is a Hamiltonian cycle of D.

Let $W = x_1x_2\dots x_k$ be a cycle or a path, where x_i's are vertices for $i \in [k]$ and $x_i \to x_{i+1}$ for $i \in [k-1]$ (Recall that $x_1 = x_k$ if W is a cycle). We use the following notation for a subpath of W:

$$W[x_i, x_j] = x_ix_{i+1}\dots x_j.$$

Let $P_1, P_2, \dots, P_q$ be paths and $C_1, C_2, \dots, C_t$ be cycles which are vertex-disjoint pairwise. If $\mathcal{F} = P_1 \cup P_2 \cup \cdots \cup P_q$ is a spanning subdigraph of D, then $\mathcal{F}$ is called a q-path factor of D. If $\mathcal{F} = C_1 \cup C_2 \cup \cdots \cup C_t$ is a spanning subdigraph of D, then $\mathcal{F}$ is called a t-cycle factor of D. If $\mathcal{F} = P_1 \cup P_2 \cup \cdots \cup P_q \cup C_1 \cup C_2 \cup \cdots \cup C_t$ is a spanning subdigraph of D, then $\mathcal{F}$ is called a q-path-cycle factor of D.

The underlying graph of a digraph D is the graph obtained by ignoring the orientations of arcs in D and deleting parallel edges. We say that D is connected if its underlying graph is connected. In this paper, we only consider connected digraphs.

A digraph $D = (V, A)$ is called strongly connected (or just strong) if there exists a path from x to y and a path from y to x in D for every choice of distinct vertices x, y of D, and D is k-arc-strong (respectively, k-strong) if $D - X$ is strong for every subset $X \subseteq A$ (respectively, $X \subseteq V$) of size at most $k - 1$. Note that a digraph with only one vertex is strong.

A strong component of a digraph D is a maximal induced subdigraph of D which is strong. For any non-strong digraph D, we can label its strong components $D_1, D_2, \dots, D_p$, $p \geq 2$, in such a way that there is no arc from D_j to D_i when $j > i$. We call $D_1, D_2, \dots, D_p$ an acyclic ordering of the strong components of D. We call D_1 the initial and D_p the terminal strong component of D.

If D is strong and $S \subset V(D)$ such that $D - S$ is not strong, then S is called a separator of D. A separator S is minimal if no proper subset of S is a separator of D.

A digraph D is semicomplete if, for every pair x, y of vertices of D, either x dominates y or y dominates x (or both). A digraph D is locally semicomplete if for every vertex x, the out-neighborhood of x induces a semicomplete digraph and the in-neighborhood of x induces a semicomplete digraph. A semicomplete digraph without a 2-cycle is a tournament and a locally semicomplete digraph without a 2-cycle is a local tournament.

A tournament is called transitive if it contains no cycle. It is easy to see that, for a transitive tournament T, there is a unique vertex ordering $v_1, v_2, \dots, v_n$ of T, such that $v_i \to v_j$ for all $1 \leq i < j \leq n$. A tournament is almost transitive if it is obtained from the transitive tournament T by reversing the arc v_1v_n.

A digraph R on r vertices is round if we can label its vertices $x_1, x_2, \dots, x_r$ so that for each i, we have $N_R^+(x_i) = \{x_{i+1}, x_{i+2}, \dots, x_{i+d_R^+(x_i)}\}$ and $N_R^-(x_i) = \{x_{i-d_R^-(x_i)}, \dots, x_{i-2}, x_{i-1}\}$ (all subscripts are taken modulo r). Note that every round digraph is locally semicomplete and a round digraph without a 2-cycle is a local tournament. If a local tournament R is round then there exists a unique (up to cyclic permutations) labeling of vertices of R which satisfies the properties in the definition. We refer to this as the round labeling of R. See Fig. 1(a) for an example of a round digraph R. Observe that the ordering $x_1, x_2, \dots, x_6$ is a round labeling of R. The second power of a cycle C_n, denoted by C_n^2, is the digraph obtained from C_n by adding the arcs $\{x_ix_{i+2} : i \in [n]\}$, where $C_n = x_1x_2\dots x_nx_1$ and subscripts are modulo n. Clearly, C_n^2 is a round digraph. See Fig. 1(b), the second power of an 8-cycle.

Let R be a digraph with vertex set $\{x_i : i \in [r]\}$, and $D_1, D_2, \dots, D_r$ be digraphs which are pairwise vertex-disjoint. Let $D = R[D_1, D_2, \dots, D_r]$ be the new digraph obtained from R by replacing x_i with D_i and adding arc from every vertex of D_i to every vertex of D_j if and only if $x_i \to x_j$ in R. If R is a round digraph and each D_i is a strong semicomplete digraph, it is easy to see that $D = R[D_1, D_2, \dots, D_r]$ is a locally semicomplete digraph. We call D a round decomposable locally semicomplete digraph and $R[D_1, D_2, \dots, D_r]$ a round decomposition of D. If a round decomposable locally semicomplete digraph $D = R[D_1, D_2, \dots, D_r]$ has no 2-cycle (i.e. the round digraph R has no 2-cycle and each D_i, $i \in [r]$ is a strong tournament or a single vertex), we say that D is a round decomposable local tournament.

Locally semicomplete digraphs were introduced in 1990 by Bang-Jensen [1]. The following theorem, due to Bang-Jensen, Guo, Gutin and Volkmann, stated a full classification of locally semicomplete digraphs.

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