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Patterns in treeshelves

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ABSTRACT

We study the distribution and the popularity of left children on sets of treeshelves avoiding a pattern of size three. (Treeshelves are ordered binary increasing trees where every child is connected to its parent by a left or a right link.) The considered patterns are sub-treeshelves, and for each such a pattern we provide exponential generating function for the corresponding distribution and popularity. Finally, we present constructive bijections between treeshelves avoiding a pattern of size three and some classes of simpler combinatorial objects.

1. Introduction and notation

The study of patterns in permutations was first introduced by Knuth [13], and continues to be an active area of research today. Recently, patterns have been studied in contexts other than permutations, see for instance [5,16] where the combinatorial class under consideration are inversion sequences, which can be seen as an alternative representation for permutations. The present paper deals with treeshelves (formally defined below) which are still another class in bijection with permutations, and patterns are sub-treeshelves contained or avoided in a similar way as consecutive patterns in permutations or in inversion sequences. More precisely, we consider the class of unrestricted treeshelves and of those avoiding a pattern of size 3 (treeshelves avoiding a pattern of size 2 collapse trivially to a singleton set). We not only enumerate these classes for any avoider of size 3, but also give bivariate generating functions with respect to the size and to the number of occurrences of a second pattern of size 2. As a byproduct we obtain the popularity (*i.e.*, the cumulative number of occurrences) among these classes of the pattern of size 2, obtaining counting sequences which are not yet recorded in Sloane's Encyclopedia of Integer Sequences [20].

Treeshelves are particular classes of binary increasing trees, considered for example in Françon's work [10] in the context of data structures for binary search methods. An *increasing tree* of size n, is an ordered rooted tree with n nodes labeled by distinct integers in $\{1, 2, \ldots, n\}$, so that the sequences of labels are increasing along all branches starting at the root (and thus, the root is labeled by 1). A *binary increasing tree* (sometimes called 0–1–2 increasing tree) is an increasing tree where every node has at most two children. Many studies (e.g., [1,2,3,6,15]) investigate binary increasing trees, but very few deal with such trees endowed with the additional property that every child (including those with no siblings) is connected to its parent by either a left or a right link. We call such a binary increasing tree *treeshelf* (or *t-shelf* for short), and its size is the number of its nodes, see Fig. 1 for a size 7 t-shelf. We denote by $\mathcal{B}_n$ the set of size n t-shelves, and $\mathcal{B}_1$ consists of the single one-node t-shelf. Often it is more convenient to represent graphically t-shelves by trees where edges are elongated so that

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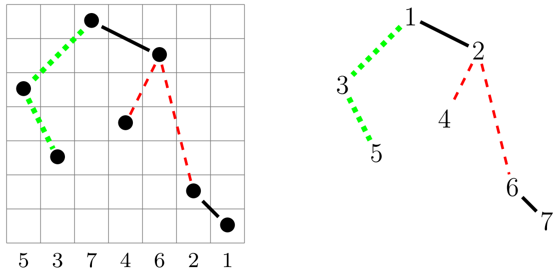

Fig. 1. The t-shelf corresponding to the permutation 5 3 7 4 6 2 1; dotted (dashed) lines correspond to an occurrence of the third (fourth) pattern in the set $\mathcal{B}_3$ listed above.

vertex i is at level i, for each i. For example, the size 3 t-shelf

is represented by , and is represented by ,

see also Fig. 1. In this representation, $\mathcal{B}_2 = \{ , \}$, and $\mathcal{B}_3 = \{ , , , , , \}$.

We denote $\cup_{n\geq 0}\mathcal{B}_n$ by $\mathcal{B}$, and $\cup_{n\geq 1}\mathcal{B}_n$ by $\mathcal{B}^\bullet$. The labeled tree rooted at the left child of the root of a t-shelf T becomes a t-shelf after appropriately relabeling its nodes, and in the following we refer to it as the *left t-shelf of* T, and similarly for the *right t-shelf of* T.

There is a bijection between $\mathcal{B}_n$ and the set of permutations of size n, and so the cardinality of $\mathcal{B}_n$ is $n!$. Indeed, to any t-shelf T in $\mathcal{B}_n$ we can uniquely associate the length n permutation $\pi = \alpha(n - r(T) + 1)\beta$, where $r(T)$ is the label of the root of T, and α (resp. β) is recursively defined from the left (resp. right) t-shelf of T (see again Fig. 1). As mentioned by Bergeron, Flajolet, and Salvy [1], this construction appears in [10] and thereafter recalled in Stanley's book [21]. Additional information (including historical notes) about binary and other families of increasing trees can be found for example in [1,4,11].

In this paper we are interested in the sets of t-shelves avoiding a pattern $P \in \mathcal{B}_3$, *i.e.*, the sets of those that do not contain any occurrence of P. The containment/avoidance of a pattern in a t-shelf can most easily be explained with examples. The avoidance of in a t-shelf T means that T does not contain any node where the label of its left child is less than that of its right child. The t-shelf in Fig. 1 contains only one pattern (illustrated by dashed lines), one pattern (dotted) and avoids the pattern .

Since the number of patterns in a t-shelf is equal to the size of the t-shelf minus the number of patterns, minus one, in the following we will consider only patterns. Moreover, an occurrence of the pattern is equivalent to that of a left child in the underlying tree of the t-shelf, we will refer to this pattern as a left child (similarly the pattern corresponds to a right child). Also, since the patterns and are equivalent by symmetry, and so are the patterns and , and the patterns and , we will consider only avoiders P in $\{ , , \}$.

T-shelves are labeled combinatorial objects, and so it is appropriate to use exponential generating functions (e.g.f.) for the enumerative analysis of them. In Section 2, for each of the avoiders P above mentioned, we consider the set $\mathcal{B}(P)$ of t-shelves avoiding P, or $\mathcal{B}^\bullet(P)$ when we restrict to non-empty t-shelves. We provide a bivariate exponential generating function for each $\mathcal{B}(P)$ with respect to the size and the number of left children, that is, function where the coefficient of $\frac{z^n y^k}{n!}$ in its series expansion is the number of t-shelves of size n having exactly k left children, and deduce the e.g.f. for $\mathcal{B}(P)$ with respect to the size. We also give the e.g.f. for the popularity of the left children among $\mathcal{B}(P)$, function where the coefficient of $\frac{z^n}{n!}$ in its series expansion is the total number of left children appearing in all size n t-shelves in $\mathcal{B}(P)$. These results are summarized in Tables 1 and 2.

Our method consists in constructing recursively the combinatorial class in question from two smaller classes, $\mathcal{A}_1$ and $\mathcal{A}_2$, using the usual labeled product $\mathcal{A}_1 \star \mathcal{A}_2$ and the boxed product $\mathcal{A}_1^{\Box} \star \mathcal{A}_2$. The boxed product $\mathcal{A}_1^{\Box} \star \mathcal{A}_2$ is a subset of $\mathcal{A}_1 \star \mathcal{A}_2$ where the smallest label appears in the $\mathcal{A}_1$ component. See [8] for more information about the boxed product and its application on labeled combinatorial structures.

Theorems 4–6 in Section 3 give constructive proofs of some results in Section 2, namely constructive bijections between: (i) t-shelves avoiding and set partitions, (ii) unordered binary increasing trees where every node of degree one has either a left or a right child and t-shelves avoiding the pattern , and (iii) unordered binary increasing trees and t-shelves avoiding the pattern .

2. T-shelves avoiding a size 3 pattern

We begin this section by considering unrestricted t-shelves, then we extend our approach to those avoiding a pattern in $\{ , , \} \subset \mathcal{B}_3$.