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Longest cycles in 4-connected graphs[☆]

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ABSTRACT

Let $\sigma_4^* = \min\{\sum_{i=1}^4 d(v_i) + |\bigcup_{i=1}^4 N(v_i)| - |\bigcap_{i=1}^4 N(v_i)| : \{v_1, v_2, v_3, v_4\}$ is an independent set of a graph G}. In this paper, we give a low bound for the length of a longest cycle in a 4-connected graph and get the following result: If G is a 4-connected graph on n vertices, then the circumference $c(G) \geq \min\{n, \sigma_4^*/2\}$. Moreover, we give graphs to show that the connectivity in our result is best possible with respect to the low bound and the low bound in our result is also best possible with respect to the connectivity.

1. Introduction

In this paper, we consider only undirected, finite and simple graphs. The notation and terminology not defined here can be found in [2]. Let $G = (V(G), E(G))$ be a graph and H be a subgraph of G. For a vertex $u \in V(G)$, the *neighborhood* of u in H is denoted by $N_H(u) = \{v \in V(H) : uv \in E(G)\}$ and the *degree* of u in H is denoted by $d_H(u) = |N_H(u)|$. For two vertices $u, v \in V(H)$, $P_H[u, v]$ denotes a longest path between u and v in the subgraph H of G. For a subset $S \subseteq V(G)$, $N_H(S)$ denotes the set of vertices in H which are adjacent to some vertices in S. If $H = G$, we use $N(u)$, $d(u)$, $P[u, v]$ and $N(S)$ in place of $N_G(u)$, $d_G(u)$, $P_G[u, v]$ and $N_G(S)$, respectively.

For a cycle C in G with a given orientation and a vertex x in C, let x^+ and x^- denote the successor and the predecessor of x in C, respectively. We set $x^{+(h+1)} = (x^{+h})^+$ and $x^{-(h+1)} = (x^{-h})^-$ for $h \geq 1$. For two vertices $x, y \in V(C)$, $C[x, y]$ or xCy denotes the subpath of C from x to y. The reverse sequence of $C[x, y]$ is denoted by $\overline{C}[y, x]$ or $y\overline{C}x$. We write $C(x, y] = C[x^+, y]$ and $C[x, y) = C[x, y^-]$. A similar notation is used for paths.

Let $S \neq \emptyset$ be a subset of $V(G)$ and $k \geq 1$ be an integer. We denote $\overline{\sigma}_k(S) = \min\{\sum_{i=1}^k d(u_i) - |\bigcap_{i=1}^k N(u_i)| : \{u_1, u_2, \ldots, u_k\}$ is an independent set of $S\}$, $\sigma_k^*(S) = \min\{\sum_{i=1}^k d(u_i) + |\bigcup_{i=1}^k N(u_i)| - |\bigcap_{i=1}^k N(u_i)| : \{u_1, u_2, \ldots, u_k\}$ is an independent set of $S\}$ and $\tilde{e}(S, \{u_1, u_2, \ldots, u_k\}) = \sum_{i=1}^k d_S(u_i) + |\bigcup_{i=1}^k N_S(u_i)| - |\bigcap_{i=1}^k N_S(u_i)|$. If $S = V(G)$, we denote $\overline{\sigma}_k = \overline{\sigma}_k(G)$ and $\sigma_k^* = \sigma_k^*(G)$, respectively. Clearly, $\sigma_k^* = \min\{\tilde{e}(G, \{u_1, u_2, \ldots, u_k\}) : \{u_1, u_2, \ldots, u_k\}$ is an independent set of $G\}$.

A graph G is *hamiltonian* if it contains a *hamiltonian cycle*, i.e. a cycle that contains all vertices of G. In 1991, Flandrin, Jung and Li gave a sufficient condition for a 2-connected graph to be hamiltonian.

Theorem 1.1 (*Flandrin, Jung and Li [4]*). *Let G be a 2-connected graph on $n \geq 3$ vertices. If $\overline{\sigma}_3 \geq n$, then G is hamiltonian.*

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The *circumference* of a graph G, denoted by $c(G)$, is the length of a longest cycle of G. Wei gave a low bound for the circumference of 3-connected graphs as follows.

Theorem 1.2 (*Wei [7]*). *If G is a 3-connected graph on n vertices, then $c(G) \geq \min\{n, \overline{\sigma}_3\}$.*

A subset S of $V(G)$ is *cyclable* in G if there is a cycle in G containing all vertices of S. Clearly, if $S = V(G)$, "S is cyclable" is equivalent to "G is hamiltonian". In 2016, Li and Zhu got the following result.

Theorem 1.3 (*Li and Zhu [6]*). *Let G be a 3-connected graph on n vertices and S be a subset of $V(G)$. If $\sigma_4^*(S) \geq 2n-1$, then S is cyclable in G.*

In this paper, we shall consider the relationship between σ_4^* and $c(G)$ and give a low bound of the circumference of 4-connected graphs.

Theorem 1.4. *If G is a 4-connected graph on n vertices, then $c(G) \geq \min\{n, \sigma_4^*/2\}$.*

We will prove Theorem 1.4 in Section 3. Now we give two examples to show that the connectivity in our result is best possible with respect to the low bound, the low bound in our result is also best possible with respect to the connectivity and our result improves Wei's theorem (Theorem 1.2).

Example 1. Let $G = 3K_1 + mK_{(n-3)/m}$ with $m \geq 4$. Clearly, G is 3-connected and $\sigma_4^* = 8(n-3)/m + 4$. But $c(G) = 3(n-3)/m + 3 < \sigma_4^*/2$. So the connectivity in our result is best possible with respect to the low bound.

Example 2. Let $G = 4K_1 + mK_{(n-4)/m}$ with $m \geq 5$. It is easy to check that G is 4-connected, $\overline{\sigma}_3 = 3(n-4)/m + 5$, $\sigma_4^* = 8(n-4)/m + 8$ and $c(G) = 4(n-4)/m + 4 = \sigma_4^*/2$. This shows that the lower bound $\sigma_4^*/2$ in our result is best possible and our result improves Theorem 1.2.

2. Preliminaries

Following Fraisse and Jung [5], for a 2-connected graph G, let $D(G)$ be a maximum integer s such that there is an (x, y)-path P of length $|V(P)| - 1 \geq s$ for any two distinct vertices x and y in G. If G is the complete graph K_n, then we set $D(G) = n - 1$. If G is connected and has cut vertices, then we set $D(G) = \max\{D(G') : G'$ is an endblock of $G\}$. If G is disconnected, then we set $D(G) = \max\{D(G') : G'$ is a component of $G\}$.

Suppose G is a 4-connected non-hamiltonian graph. Let C be a longest cycle of G with a given orientation such that $D(G - C) = \max\{D(G - C') : C'$ is a longest cycle of $G\}$ and H be a component of $G - C$ such that $D(H) = D(G - C)$. Suppose $N_C(H) = \{x_1, x_2, \ldots, x_t\}$ $(t \geq 4)$ and the occurrence of $x_1, x_2, \ldots, x_t$ on C agrees with the given orientation of C. A vertex $u \in x_i^+ C x_{i+1}^-$ is *insertible* if there are vertices $v, v^+ \in x_{i+1} C x_i$ such that $uv, uv^+ \in E(G)$. We denote the first non-insertible vertex occurring on $x_i^+ C x_{i+1}^-$ by u_i. Let $U_i = x_i^+ C u_i$ and $T = \{u_1, u_2, \ldots, u_t\}$. Our proof of Theorem 1.4 is based on the following lemmas.

Lemma 2.1 (*Ainouche [1], Broersma and Lu [3]*). *(1) u_i exists;*
(2) $u_i \notin N(H)$ for any $u_i \in T$;
(3) There is no path whose internal vertices (if any) in $G - C$ join U_i and U_j;
(4) For any $u_i, u_j \in T$, $u_j w \notin E(G)$ whenever $u_i w^- \in E(G)$ and $w \in u_j^+ C x_i$.

Lemma 2.2 (*Fraisse and Jung [5]*). *Let G be a connected graph. If G is 2-connected and not complete, then there are non-adjacent vertices v_1, v_2 in G such that v_i is not a cut vertex of G and $d(v_i) \leq D(G)$ for $i = 1, 2$. If G is a complete graph, then $d(v) \leq D(G)$ for all $v \in V(G)$. If G has cut vertices, let G_1 and G_2 be distinct blocks of G, then for any non-cut vertex $v_i \in V(G_i)(i = 1, 2)$, $|P[v_1, v_2]| \geq D(G_1) + D(G_2) + 1$.*

Lemma 2.3 (*Wei [7]*). *Let $u, v \in V(H)$ and $x_i u \in E(G)$ for some $1 \leq i \leq t$. We have*
(i) there is no path connecting u_i and a vertex $y \in V(C(x_i, u_i))$ with internal vertices in $V(G - C)$;
(ii) $u_i^{+h} v \notin E(G)$ for any $h \leq s$;
(iii) if $x_j \in N_C(v)$, $j \neq i$, then there is no any path connecting u_i^{+h} and u_j with internal vertices (if any) in $V(G - C)$ for any $h \leq s$.
Here s denotes the length of a longest path between u and v in H, in particular, $s = 0$ when $v = u$.

Lemma 2.4. *Let $P = v_1 v_2 \ldots v_p$ be a path and w_1, w_2, w_3, w_4 be four vertices not in $V(P)$ such that $N_P(w_i) \cap N_P(w_i)^+ = \emptyset$ for every $i \in \{2, 3\}$ and $N_P(w_i) \cap N_P(w_j)^+ = \emptyset$ for $1 \leq i < j \leq 4$. Then $\tilde{e}(P, \{w_1, w_2, w_3, w_4\}) \leq 2|V(P)| + 2$. Furthermore, if $v_1 \notin N_P(\{w_1, w_i\})$ for some $i \in \{2, 3, 4\}$, then $\tilde{e}(P, \{w_1, w_2, w_3, w_4\}) \leq 2|V(P)| + 1$; if $v_1 \notin N_P(\{w_1, w_2, w_i\})$ for some $i \in \{3, 4\}$, then $\tilde{e}(P, \{w_1, w_2, w_3, w_4\}) \leq 2|V(P)|$.*