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On the radius and the attachment number of tetravalent half-arc-transitive graphs

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ABSTRACT

In this paper, we study the relationship between the radius r and the attachment number a of a tetravalent graph admitting a half-arc-transitive group of automorphisms. These two parameters were first introduced in Marušič (1998), where among other things it was proved that a always divides $2r$. Intrigued by the empirical data from the census (Potočnik et al., 2015) of all such graphs of order up to 1000 we pose the question of whether all examples for which a does not divide r are arc-transitive. We prove that the answer to this question is positive in the case when a is twice an odd number. In addition, we completely characterise the tetravalent graphs admitting a half-arc-transitive group with $r = 3$ and $a = 2$, and prove that they arise as non-sectional split 2-fold covers of line graphs of 2-arc-transitive cubic graphs.

1. Introduction

This paper stems from our research of finite simple connected tetravalent graphs that admit a group of automorphisms acting transitively on vertices and edges but not on the arcs of the graph; such groups of automorphisms are said to be *half-arc-transitive*. Observe that the full automorphism group $\mathrm{Aut}(\Gamma)$ of such a graph Γ is then either arc-transitive or itself half-arc-transitive. In the latter case the graph Γ is called *half-arc-transitive*.

Tetravalent graphs admitting a half-arc-transitive group of automorphisms are surprisingly rich combinatorial objects with connections to several other areas of mathematics (see, for example, [1,9–12,16,18]). One of the most fruitful tools for analysing the structure of a tetravalent graph Γ admitting a half-arc-transitive group G is to study a certain G-invariant decomposition of the edge set $E(\Gamma)$ of Γ into the *G-alternating cycles* of some even length $2r$; the parameter r is then called the *G-radius* and denoted $\mathrm{rad}_G(\Gamma)$ (see Section 2 for more detailed definitions). Since G is edge-transitive and the decomposition into G-alternating cycles is G-invariant, any two intersecting G-alternating cycles meet in the same number of vertices; this number is then called the *attachment number* and denoted $\mathrm{att}_G(\Gamma)$. When $G = \mathrm{Aut}(\Gamma)$ the subscript G will be omitted in the above notation.

It is well known and easy to see that $\mathrm{att}_G(\Gamma)$ divides $2\,\mathrm{rad}_G(\Gamma)$. However, for all known tetravalent half-arc-transitive graphs the attachment number in fact divides the radius. This brings us to the following question that we would like to propose and address in this paper:

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Question 1. *Is it true that the attachment number* $\mathrm{att}(\Gamma)$ *of an arbitrary tetravalent half-arc-transitive graph* Γ *divides the radius* $\mathrm{rad}(\Gamma)$*?*

By checking the complete list of all tetravalent half-arc-transitive graphs on up to 1000 vertices (see [15]), we see the that answer to the above question is affirmative for the graphs in that range. Further, as was proved in [14, Theorem 1.2], the question has an affirmative answer in the case $\mathrm{att}(\Gamma) = 2$. In Section 3, we generalise this result by proving the following theorem.

Theorem 2. *Let* Γ *be a tetravalent half-arc-transitive graph. If its radius* $\mathrm{rad}(\Gamma)$ *is odd, then* $\mathrm{att}(\Gamma)$ *divides* $\mathrm{rad}(\Gamma)$*. Consequently, if* $\mathrm{att}(\Gamma)$ *is not divisible by* 4*, then* $\mathrm{att}(\Gamma)$ *divides* $\mathrm{rad}(\Gamma)$*.*

As a consequence of our second main result (Theorem 3) we see that, in contrast to Theorem 2, there exist infinitely many arc-transitive tetravalent graphs Γ admitting a half-arc-transitive group G with $\mathrm{rad}_G(\Gamma) = 3$ and $\mathrm{att}_G(\Gamma) = 2$. In fact, in Section 2, we characterise these graphs completely and prove the following theorem (see Section 2.2 for the definition of the dart graph).

Theorem 3. *Let* Γ *be a connected tetravalent graph. Then* Γ *is* G*-half-arc-transitive for some* $G \leq \mathrm{Aut}(\Gamma)$ *with* $\mathrm{rad}_G(\Gamma) = 3$ *and* $\mathrm{att}_G(\Gamma) = 2$ *if and only if* Γ *is the dart graph of some* 2*-arc-transitive cubic graph.*

The third main result of this paper, stemming from our analysis of the situation described by Theorem 3, reveals a surprising connection to the theory of covering projections of graphs. This theory has become one of the central tools in the study of symmetries of graphs. A particularly thrilling development started with the seminal work of Malnič, Nedela and Škoviera [5] who analysed the condition under which a given automorphism group of the base graph lifts along the covering projection. Recently, the question of determining the structure of the lifted group received a lot of attention (see [2,6,7]).

To be more precise, let $\wp : \tilde{\Gamma} \to \Gamma$ be a covering projection of connected graphs and let $\mathrm{CT}(\wp)$ be the corresponding group of covering transformations (see [5], for example, for the definitions pertaining to the theory of graph covers). Furthermore, let $G \leq \mathrm{Aut}(\Gamma)$ be a subgroup that lifts along $\wp$. Then the lifted group $\tilde{G}$ is an extension of $\mathrm{CT}(\wp)$ by G. If this extension is split then the covering projection $\wp$ is called *G-split*. The most natural way in which this can occur is that there exists a complement $\bar{G}$ of $\mathrm{CT}(\wp)$ in $\tilde{G}$ and a $\bar{G}$-invariant subset S of $V(\tilde{\Gamma})$, that intersects each fibre of $\wp$ in exactly one vertex. In such a case we say that S is a *section* for $\bar{G}$ and that $\bar{G}$ is a *sectional* complement of $\mathrm{CT}(\wp)$. Split covering projections without any sectional complement are called *non-sectional*. These turn out to be rather elusive and hard to analyse. To the best of our knowledge, the only known infinite family of non-sectional split covers was presented in [2, Section 4]. This family of non-sectional split covers involves cubic arc-transitive graphs of extremely large order.

In this paper we show that each connected tetravalent graph Γ admitting a half-arc-transitive group G of automorphisms such that $\mathrm{att}_G(\Gamma) = 2$ and $\mathrm{rad}_G(\Gamma) = 3$ is a 2-fold cover of the line graph of a cubic 2-arc-transitive graph, and that in the case when Γ is not bipartite the corresponding covering projection is non-sectional. This thus provides a new and rather simple infinite family of the somewhat mysterious case of non-sectional split covering projections (see Section 4 for more details).

2. Half-arc-transitive group actions on graphs

In the next two paragraphs we briefly review some concepts and results pertaining half-arc-transitive group actions on tetravalent graphs that we shall need in the remainder of this section. For more details see [8], where most of these notions were introduced.

A tetravalent graph Γ admitting a *half-arc-transitive* (that is vertex- and edge- but not arc-transitive) group of automorphisms G is said to be *G-half-arc-transitive*. The action of G induces two paired orientations of the edges of Γ and for any one of them each vertex of Γ is the head of two and the tail of the other two of its incident edges. (The fact that the edge uv is oriented from u to v will be denoted by $u \to v$.) A cycle of Γ for which every two consecutive edges either have a common head or common tail with respect to this orientation is called a *G-alternating cycle*. Since the action of G is vertex- and edge-transitive all of the G-alternating cycles have the same even length $2\mathrm{rad}_G(\Gamma)$ and any two non-disjoint G-alternating cycles intersect in the same number $\mathrm{att}_G(\Gamma)$ of vertices. These intersections, called the *G-attachment sets*, form an imprimitivity block system for the group G. The numbers $\mathrm{rad}_G(\Gamma)$ and $\mathrm{att}_G(\Gamma)$ are called the *G-radius* and *G-attachment number* of Γ, respectively. If $G = \mathrm{Aut}(\Gamma)$ we suppress the prefix and subscript $\mathrm{Aut}(\Gamma)$ in all of the above definitions.

It was shown in [8, Proposition 2.4] that a tetravalent G-half-arc-transitive graph Γ has at least three G-alternating cycles unless $\mathrm{att}_G(\Gamma) = 2\mathrm{rad}_G(\Gamma)$ in which case Γ is isomorphic to a particular Cayley graph of a cyclic group (and is thus arc-transitive). Moreover, in the case that Γ has at least three G-alternating cycles, $\mathrm{att}_G(\Gamma) \leq \mathrm{rad}_G(\Gamma)$ holds and $\mathrm{att}_G(\Gamma)$ divides $2\mathrm{rad}_G(\Gamma)$. In addition, the restriction of the action of G to any G-alternating cycle is isomorphic to the dihedral group of order $2\mathrm{rad}_G(\Gamma)$ (or to the Klein 4-group in the case of $\mathrm{rad}_G(\Gamma) = 2$) with the cyclic subgroup of order $\mathrm{rad}_G(\Gamma)$ being the subgroup generated by a two-step rotation of the G-alternating cycle in question. In addition, if $C = (v_0, v_1, \ldots, v_{2r-1})$ is a G-alternating cycle of Γ with $r = \mathrm{rad}_G(\Gamma)$ and C' is the other G-alternating cycle of Γ containing v_0 then $C \cap C' = \{v_{i\ell} : 0 \leq i < a\}$ where $a = \mathrm{att}_G(\Gamma)$ and $\ell = 2r/a$ (see [8, Proposition 2.6] and [13, Proposition 3.4]).

As mentioned in the Introduction one of the goals of this paper is to characterise the tetravalent G-half-arc-transitive graphs Γ with $\mathrm{rad}_G(\Gamma) = 3$ and $\mathrm{att}_G(\Gamma) = 2$. The bijective correspondence between such graphs and 2-arc-transitive cubic graphs (see Theorem 3) is given via two pairwise inverse constructions: the *graph of alternating cycles* construction and the *dart graph* construction. We first define the former.

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