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Note

On constructing normal and non-normal Cayley graphs[☆]

Yian Xu

Centre for the Mathematics of Symmetry and Computation, School of Mathematics and Statistics, The University of Western Australia, Australia

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ABSTRACT

Bamberg and Giudici (2011) showed that the point graphs of certain generalised quadrangles of order $(q-1, q+1)$, where $q = p^k$ is a prime power with $p \geq 5$, are both normal and non-normal Cayley graphs for two isomorphic groups. We call these graphs BG-graphs. In this paper, we show that the Cayley graphs obtained from a finite number of BG-graphs by Cartesian product, direct product, and strong product also possess the property of being normal and non-normal Cayley graphs for two isomorphic groups.

1. Introduction

Let G be a finite group, and S be a subset of G such that S does not contain the identity of G and $S = S^{-1} = \{s^{-1} | s \in S\}$. The *Cayley graph* $\Gamma = Cay(G, S)$ is defined to have vertex set $V(\Gamma) = G$, and edge set $E(\Gamma) = \{\{g, sg\} | s \in S\}$. Let $Aut(\Gamma)$ denote the automorphism group of Γ. For each $g \in G$, define a map $\hat{g} : G \to G$ by the right multiplication of g on G as below:

$$\hat{g} : x \to xg, \ for\ x \in G.$$

Then $\hat{g}$ is an automorphism of Γ. It follows from the definition that the group $\hat{G} = \{\hat{g} | g \in G\}$ is a subgroup of $Aut(\Gamma)$ and acts regularly on $V(\Gamma)$. It is well known that a graph is a Cayley graph if and only if its automorphism group contains a subgroup that acts regularly on the vertex set of the graph (see [9], Theorem 16.3 in [2]).

We say Γ is a *normal Cayley graph for* G (Xu, [10]) (or normal) if $\hat{G} \trianglelefteq Aut(\Gamma)$, otherwise we say Γ is a *non-normal Cayley graph for* G (or non-normal). In [10], Xu showed that except for $Z_4 \times Z_2$ and $Q_8 \times Z_2^m$ with $m \geq 0$, each finite group has at least one normal Cayley graph. Years later, Feng and Dobson [1] proposed a question asking if it is possible for a Cayley graph to be both normal and non-normal for two isomorphic regular groups, that is, can $Aut(\Gamma)$ contain a normal regular subgroup G and a non-normal regular subgroup isomorphic to G.

We call a graph that is a normal and non-normal Cayley graph for isomorphic regular groups an *NNN-graph* (or say the graph is *NNN*). Very few NNN-graphs are known. Giudici and Smith [3] constructed a strongly regular Cayley graph for Z_6^2 and showed that such a graph is NNN. Royle [8] proved that the halved folded 8-cube is an NNN-graph. In [1], Bamberg and Giudici showed that the point graphs of a particular family of generalised quadrangles denoted Q^x (we give more details later) are NNN-graphs. In this paper we call the point graph of a Q^x a *BG-graph*.

To construct new NNN-graphs, a natural idea is to apply graph products to some known NNN-graphs. There are three standard graph products that have been extensively studied throughout the literature: the Cartesian product, the direct

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E-mail address: yian.xu@research.uwa.edu.au.

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product, and the strong product. We will give their definitions in Section 2; interested readers are referred to [5] for more details. In this paper, we show that the standard graph products of finitely many BG-graphs are still NNN-graphs.

Theorem 1.1. *Let $\Gamma_1, \dots, \Gamma_t$ be BG-graphs. Then the graph obtained by taking one of the three standard products of $\Gamma_1, \dots, \Gamma_t$ is an NNN-graph.*

In Section 2, we will present some definitions and known results used in this paper. Section 3 is devoted to investigating the products of prime NNN-graphs. The proof of Theorem 1.1 is given in Section 4.

A *generalised quadrangle* Q of *order* (s, t) is an incidence structure $(\mathcal{P}, \mathcal{L}, I)$, with a set of points $\mathcal{P}$, a set of lines $\mathcal{L}$, and an *incidence relation* $I \subseteq \mathcal{P} \times \mathcal{L}$, satisfying the following $GQ-$ axioms:

1. There are exactly $s + 1$ points on every line, and there is at most one point on two distinct lines.
2. Each point is on exactly $t + 1$ lines, and there is at most one line through two distinct points.
3. For each point x not on a line L, there is a unique line M and a unique point y such that x is on M, and y is on M and L.

The points on the same line are said to be *collinear*, and two lines that pass through a common point are said to be *concurrent*. A generalised quadrangle is said to be *thick* if $s, t \geq 2$. The *point graph* of a generalised quadrangle Q is the graph having $\mathcal{P}$ as its vertex set, and two vertices x and y are adjacent if and only if they are collinear in Q. In Section 4 we will show that the point graph of a thick generalised quadrangle has the same automorphism group as the generalised quadrangle.

Most of the known generalised quadrangles are *elation generalised quadrangles*. The automorphism group of such a generalised quadrangle contains a subgroup of automorphisms G that fixes a point x and each line through x, and acts regularly on the points not collinear with x. One class of such geometries are the symplectic generalised quadrangles $W(3, q)$ of order (q, q). Let $Q = W(3, q)$. In [7], Payne introduced a method to construct a new generalised quadrangle Q^x from Q. The points of Q^x are those in Q not collinear with a point x. The automorphism group of Q^x contains $\mathrm{P}\Gamma\mathrm{Sp}(4, q)_x$, and $\mathrm{P}\Gamma\mathrm{Sp}(4, q)_x$ is the automorphism group of Q^x when $q \geq 5$ (see [4]).

2. Preliminaries

In this section we introduce three standard graph products and some known results.

Let Γ_1 be a finite simple graph with vertex set V_1 and edge set E_1, and Γ_2 be a finite simple graph with vertex set V_2 and edge set E_2. The *Cartesian product* $\Gamma_1 \square \Gamma_2$ is the graph with vertex set $V_1 \times V_2$ such that $\{(a_1, a_2), (b_1, b_2)\}$ is an edge if and only if either $\{a_1, b_1\} \in E_1$ and $a_2 = b_2$, or $\{a_2, b_2\} \in E_2$ and $a_1 = b_1$. Many interesting classes of graphs are Cartesian products of finite graphs, for example, hypercubes and Hamming graphs (see [6]). The *direct product* $\Gamma_1 \times \Gamma_2$ is the graph with vertex set $V_1 \times V_2$ such that $\{(a_1, a_2), (b_1, b_2)\}$ is an edge if and only if $\{a_1, b_1\} \in E_1$ and $\{a_2, b_2\} \in E_2$. The *strong product* $\Gamma_1 \boxtimes \Gamma_2$ is the graph with vertex set $V_1 \times V_2$ such that $\{(a_1, a_2), (b_1, b_2)\}$ is an edge if and only if $\{a_i, b_i\} \in E_i$ for $1 \leq i \leq 2$, or $a_1 = b_1$ and $\{a_2, b_2\} \in E_2$, or $\{a_1, b_1\} \in E_1$ and $a_2 = b_2$. All these standard graph products are commutative and associative, and for each of them we call Γ_1 and Γ_2 the *factors* of the product. Graphs that are not representable as any of these three standard graph products of nontrivial graphs (that is of graphs with at least two vertices) are said to be *prime*.

Let Γ be a graph, and v be a vertex of Γ. The *neighbourhood* of v is $N(v) = \{u | \{u, v\} \in E(\Gamma)\}$, and the *closed neighbourhood* of v is $N[v] = N(v) \cup \{v\}$. Let x and y be two vertices of Γ. We say x and y are in *relation R* if $N(x) = N(y)$, and say x and y are in *relation S* if $N[x] = N[y]$. It is easy to show that both R and S are equivalence relations on $V(\Gamma)$. A graph is said to be *R-thin* if each of its R-equivalence classes contains just one vertex, and is said to be *S-thin* if each of its S-equivalence classes contains just one vertex.

Lemma 2.1. *If Γ_1 and Γ_2 are R-thin graphs, then $\Gamma_1 \times \Gamma_2$ is R-thin; if Γ_1 and Γ_2 are S-thin graphs, then $\Gamma_1 \boxtimes \Gamma_2$ is S-thin.*

Proof. Let $\Sigma = \Gamma_1 \times \Gamma_2$. Let $x = (u_1, v_1)$ and $y = (u_2, v_2)$ be two distinct vertices of Σ. Then

$$N_\Sigma(x) = \{(w_1, z_1) | w_1 \in N_{\Gamma_1}(u_1), z_1 \in N_{\Gamma_2}(v_1)\},$$

$$N_\Sigma(y) = \{(w_2, z_2) | w_2 \in N_{\Gamma_1}(u_2), z_2 \in N_{\Gamma_2}(v_2)\}.$$

Suppose $N_\Sigma(x) = N_\Sigma(y)$. Then $N_{\Gamma_1}(u_1) = N_{\Gamma_1}(u_2)$ and $N_{\Gamma_2}(v_1) = N_{\Gamma_2}(v_2)$, which is a contradiction as Γ_1 and Γ_2 are R-thin. Thus $\Gamma_1 \times \Gamma_2$ is R-thin. The proof for $\Gamma_1 \boxtimes \Gamma_2$ being S-thin is similar. ■

Corollary 2.2. *The direct product of finitely many R-thin graphs is R-thin, and the strong product of finitely many S-thin graphs is S-thin.*

We finish this section with a well-known result on the automorphism group of graph products; readers interested in the proof of this theorem are referred to [5, Theorem 7.16 & Theorem 8.18] and [6, Theorem 15.5].

Theorem 2.3 ([5,6]). *Let $\Gamma_1, \dots, \Gamma_t$ be prime graphs. Let*

$$G = (Aut(\Gamma_{k_1}) \wr S_{n_1}) \times \cdots \times (Aut(\Gamma_{k_r}) \wr S_{n_r}), \quad (1)$$

where $\sum_{i=1}^{r} n_i = t$ and n_i is the number of factors isomorphic to Γ_{k_i}. Then

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