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Decomposability index of tournaments

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ABSTRACT

Given a tournament T, a module of T is a subset X of $V(T)$ such that for $x, y \in X$ and $v \in V(T) \setminus X$, $(x, v) \in A(T)$ if and only if $(y, v) \in A(T)$. The trivial modules of T are $\emptyset$, $\{u\}$ $(u \in V(T))$ and $V(T)$. The tournament T is indecomposable if all its modules are trivial; otherwise it is decomposable. The decomposability index of T, denoted by $\delta(T)$, is the smallest number of arcs of T that must be reversed to make T indecomposable. For $n \geq 5$, let $\delta(n)$ be the maximum of $\delta(T)$ over the tournaments T with n vertices. We prove that $\left\lceil \frac{n+1}{4} \right\rceil \leq \delta(n) \leq \left\lceil \frac{n-1}{3} \right\rceil$ and that the lower bound is reached by the transitive tournaments.

1. Introduction

Arc reversals in tournaments have been studied under different aspects: algebraic, combinatorial, algorithmic, etc. The operation of reversing a single arc allows, by successive iterations, to reverse any set of arcs. The same observation takes place when a reversal operation consists of reversing all the arcs of a subtournament. But this observation becomes invalid for other types of reversal operations involving tighter rules such as switching (or pushing) operation [2,14,17], cycle reversal operation [13,19], etc. All these reversal types and others were investigated by several authors under different considerations.

The general question is the following. What can we say about the minimum number of arcs of a tournament whose reversal results in a tournament satisfying a certain property (transitivity, reducibility [16], decomposability [18], etc.)? This number gives a measure for the negation of the given property, and its determination may involve a hard decision problem. For example, computing this number is NP-hard when the property is transitivity [1,7], in which case we recognize the well-known feedback arc set problem for tournaments. This problem is well-studied from the combinatorial and algorithmic points of view [3,5,8,12]. In [4], the authors investigated another variant of this problem by allowing to reverse, at one go, all the arcs of a subtournament.

When the property is decomposability, the problem was introduced by V. Müller and J. Pelant [18] because of its relation with strongly homogeneous tournaments (also called doubly regular tournaments) whose existence is equivalent to that of skew Hadamard matrices [6]. More precisely, the indecomposability index of a tournament T (called arrow-simplicity in [18]) is the least integer $s(T)$ such that there exist $s(T)$ arcs of T whose reversal makes T decomposable. Obviously $s(T) \leq \frac{1}{2}(|T|-1)$, where $|T|$ is the number of vertices of T. The authors prove that $s(T) = \frac{1}{2}(|T| - 1)$ if and only if the tournament T is strongly homogeneous. In other words, the strongly homogeneous tournaments are precisely the tournaments with maximal indecomposability. In a similar study, S.J. Kirkland [16] established that every tournament T can be made not strongly connected by reversing at most $\left\lfloor \frac{|T|-1}{2} \right\rfloor$ arcs, and that this bound is optimal. He also established that the regular and almost regular tournaments are the tournaments with maximal strong connectivity.

In this paper, we consider the negation of the property considered in [18], that is, indecomposability. Given a tournament T with at least five vertices, the decomposability index of T, denoted by $\delta(T)$, is the least integer m, for which there exist m

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arcs of T whose reversal makes T indecomposable. The decomposability index $\delta(T)$ can be interpreted as the distance of T to the set of indecomposable tournaments on the same vertex set as T. For $n \geq 5$, let $\delta(n)$ be the maximum of $\delta(T)$ over the tournaments T with n vertices. We prove that $\lceil \frac{n+1}{4} \rceil \leq \delta(n) \leq \lceil \frac{n-1}{3} \rceil$ and that the lower bound is reached by the transitive tournaments (see Theorem 2.1).

2. Preliminaries

A *tournament* T consists of a finite set $V(T)$ of *vertices* together with a set $A(T)$ of ordered pairs of distinct vertices, called *arcs*, such that for all $x \neq y \in V(T)$, $(x, y) \in A(T)$ if and only if $(y, x) \notin A(T)$. Such a tournament is denoted by (V(T), A(T)). The *cardinality* of T, denoted by $|T|$, is that of $V(T)$. Given a tournament T, for each subset X of $V(T)$, the tournament $(X, A(T) \cap (X \times X))$ is denoted by $T[X]$. A *subtournament* of T is a tournament $T[X]$ for some $X \subseteq V(T)$. For $X \subseteq V(T)$ (resp. $x \in V(T)$), the subtournament $T[V(T) \setminus X]$ (resp. $T[V(T) \setminus \{x\}]$) is denoted by $T - X$ (resp. $T - x$). Two tournaments T and T' are *isomorphic*, which is denoted by $T \simeq T'$, if there exists an *isomorphism* from T onto T', that is, a bijection f from $V(T)$ onto $V(T')$ such that for all $x, y \in V(T)$, $(x, y) \in A(T)$ if and only if $(f(x), f(y)) \in A(T')$.

Let T be a tournament. For distinct $x, y \in V(T)$, we use the notation $x \longrightarrow y$ to signify that $(x, y) \in A(T)$. Similarly, for $x \in V(T)$ and $Y \subseteq V(T)$, the notation $x \longrightarrow Y$ (resp. $Y \longrightarrow x$) means that $x \longrightarrow y$ (resp. $y \longrightarrow x$) for all $y \in Y$. For $X, Y \subseteq V(T)$, $X \longrightarrow Y$ signifies that for every $x \in X$, $x \longrightarrow Y$. Given a vertex x of the tournament T, the *outneighbours* (resp. *inneighbours*) of x are the elements of $N_T^+(x)$ (resp. $N_T^-(x)$), where $N_T^+(x) = \{y \in V(T) : x \longrightarrow y\}$ and $N_T^-(x) = \{y \in V(T) : y \longrightarrow x\}$. Thus, $V(T) \setminus \{x\} = N_T^+(x) \cup N_T^-(x)$ and $N_T^-(x) \longrightarrow x \longrightarrow N_T^+(x)$. The *outdegree* $d_T^+(x)$ of the vertex x is the number of its outneighbours, and the *indegree* $d_T^-(x)$ of x is the number of its inneighbours: $d_T^+(x) = |N_T^+(x)|$ and $d_T^-(x) = |N_T^-(x)|$. For example, up to isomorphism, the tournament $C_3 = (\{0, 1, 2\}, \{(0, 1), (1, 2), (2, 0)\})$ is the unique tournament with three vertices having the same outdegree.

A tournament T is *transitive* if it satisfies one of the following equivalent assertions.

- For every $x, y, z \in V(T)$, if $x \longrightarrow y \longrightarrow z$, then $x \longrightarrow z$.
- The tournament T does not admit a subtournament which is isomorphic to C_3.
- We have $\{d_T^+(v) : v \in V(T)\} = \{0, \ldots, |T| - 1\}$.

Let $n \in \mathbb{N}$. We denote by $\underline{n}$ the transitive tournament whose vertex set is $\{0, \ldots, n-1\}$ and whose arcs are the ordered pairs (i, j) such that $0 \leq i < j \leq n - 1$. Up to isomorphism, $\underline{n}$ is the unique transitive tournament with n vertices.

Now, we present the main notion. Given a tournament T, a subset I of $V(T)$ is a *module* [21] (or a *clan* [10] or an *interval* [15]) of T provided that for all $x \in V(T) \setminus I$, $x \longrightarrow I$ or $I \longrightarrow x$. For example, $\emptyset$, $\{x\}$, where $x \in V(T)$, and $V(T)$ are modules of T, called *trivial* modules. A tournament is *indecomposable* [15,20] (or *prime* [9] or *primitive* [10] or *simple* [11,18]) if all its modules are trivial; otherwise, it is *decomposable*. For example, the tournaments T with at most two vertices are indecomposable. The tournaments $\underline{3}$ and C_3 are, up to isomorphism, the only tournaments with three vertices. The tournament C_3 is indecomposable, whereas $\underline{3}$ is decomposable. Up to isomorphism, there are four tournaments with four vertices, all of them are decomposable. For $n \geq 5$, the tournament $Q_n = (\{0, \ldots, n-1\}, \{(i, j) \in \{0, \ldots, n-1\}^2 : i < j - 1 \text{ or } i = j + 1\})$ is indecomposable. With each tournament T associate its *dual* $T^\star$ defined by $V(T^\star) = V(T)$ and $A(T^\star) = \{(x, y) : (y, x) \in A(T)\}$. Notice that a tournament T and its dual share the same modules. In particular, T is indecomposable if and only if $T^\star$ is.

Let T be a tournament. An *inversion* of an arc $a = (x, y) \in A(T)$ consists of replacing the arc a by $a^\star$ in $A(T)$, where $a^\star = (y, x)$. The tournament obtained from T after reversing the arc a is denoted by $\mathrm{Inv}(T, a)$. Thus $\mathrm{Inv}(T, a) = (V(T), (A(T) \setminus \{a\}) \cup \{a^\star\})$. More generally, for $B \subseteq A(T)$, we denote by $\mathrm{Inv}(T, B)$ the tournament obtained from T after reversing all the arcs of B, that is $\mathrm{Inv}(T, B) = (V(T), (A(T) \setminus B) \cup B^\star)$, where $B^\star = \{b^\star : b \in B\}$. For example, $\mathrm{Inv}(T, A(T)) = T^\star$ and $Q_n = \mathrm{Inv}(\underline{n}, \{(i, i+1) : 0 \leq i \leq n-2\})$. Notice the following two facts.

- Given two tournaments T and T', we have $T' = \mathrm{Inv}(T, A(T) \setminus A(T'))$.
- For each $n \geq 5$, there exist indecomposable tournaments with n vertices.

Given a tournament T with $|T| \geq 5$, the previous facts allow us to define the *decomposability index* of T, denoted by $\delta(T)$, as the least integer m for which there exists $B \subseteq A(T)$ such that $|B| = m$ and $\mathrm{Inv}(T, B)$ is indecomposable. Notice that isomorphic tournaments have the same decomposability index. The same remark holds for a tournament and its dual. For $n \geq 5$, let $\delta(n)$ be the maximum of $\delta(T)$ over the tournaments T with n vertices. The following theorem is the main result of this paper.

Theorem 2.1. *For every integer $n \geq 5$, we have $\delta(\underline{n}) = \lceil \frac{n+1}{4} \rceil$ and $\lceil \frac{n+1}{4} \rceil \leq \delta(n) \leq \lceil \frac{n-1}{3} \rceil$.*

The paper is organized as follows. Some required results on indecomposable tournaments are presented in Section 3. In Section 4, we compute the decomposability index of transitive tournaments. In Section 5, we establish some upward hereditary properties of the decomposability index which allow to prove Theorem 2.1. Some related questions and perspectives are discussed in Section 6.