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Hamiltonicity of edge chromatic critical graphs

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ABSTRACT

Vizing conjectured that every edge chromatic critical graph contains a 2-factor. Believing that stronger properties hold for this class of graphs, Luo and Zhao (2013) showed that every edge chromatic critical graph of order n with maximum degree at least $\frac{6n}{7}$ is Hamiltonian. Furthermore, Luo et al. (2016) proved that every edge chromatic critical graph of order n with maximum degree at least $\frac{4n}{5}$ is Hamiltonian. In this paper, we prove that every edge chromatic critical graph of order n with maximum degree at least $\frac{3n}{4}$ is Hamiltonian. Our approach is inspired by the recent development of Kierstead path and Tashkinov tree techniques for multigraphs.

1. Introduction

We consider only simple graphs unless these specified as multigraphs. We refer to [2] for notation and terminology not defined in this paper. Let G be a graph. For a $v \in V(G)$, we use $N_G(v)$ and $d_G(v)$ to denote the neighborhood and the degree of v in G, respectively. If there is no confusion, we will use $N(v)$ and $d(v)$ to denote $N_G(v)$ and $d_G(v)$, respectively. Denote by $\delta(G)$ and $\Delta(G)$ the minimum and maximum degree of G, respectively. For a nonnegative integer d, let $V_d(G)$ and $V_{\geq d}(G)$ denote the set of vertices in $V(G)$ with degree d and at least d, respectively. In this paper, we call a vertex with degree j a *j-vertex*. For a path P and $u, v \in V(P)$, let uPv denote the subpath of P from u to v. We use $\alpha(G)$ and $c(G)$ to denote the independence number and circumference of G, respectively.

An *edge-k-coloring* of a graph G is an assignment of a color to each edge of G in such a way that every two adjacent edges are colored differently and at most k different colors are used. By default, we assume in this paper that the colors are elements of $\{1, 2, \ldots, k\}$. Denote by $\mathcal{C}^k(G)$ the set of all edge-k-colorings of a graph G. The *chromatic index* of a graph G, denoted by $\chi'(G)$, is the minimum positive integer k with $\mathcal{C}^k(G) \neq \emptyset$. Clearly, $\chi'(G) \geq \Delta(G)$. Vizing [14] on the other hand proved that $\chi'(G) \leq \Delta + 1$. This leads to a classification of graphs into two classes: A graph G is of *class I* if $\chi'(G) = \Delta(G)$ and of *class II* if $\chi'(G) = \Delta(G) + 1$. An edge e of a graph G is a *critical edge* if $\chi'(G - e) = \chi'(G) - 1$. A *$\Delta$-critical graph* is a critical class II graph with maximum degree Δ such that every edge is critical. It is of interest to understand the structures of Δ-critical graphs.

In 1965, Vizing [14] conjectured that every Δ-critical graph contains a 2-factor, which is named Vizing's 2-factor conjecture. In 1968, Vizing [16] proposed a weaker conjecture, named Vizing's independence number conjecture, that $\alpha(G) \leq \frac{n}{2}$ for every Δ-critical graph of order n. Vizing's independence number conjecture was verified by Luo and Zhao [10] for Δ-critical graphs of order n with $\Delta \geq \frac{n}{2}$, and by Grunewald and Steffen [7] for Δ-critical graphs with many edges, including all overfull graphs. Chen and Shan [4] verified Vizing's 2-factor conjecture for Δ-critical graphs of order n with $\Delta \geq \frac{n}{2}$.

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Circumferences of Δ-critical graphs have been a subject of studies. Vizing [14] showed that $c(G) \geq \Delta + 1$ for every Δ-critical graph. Fiorini and Wilson [12] showed that $c(G) \geq 2(\log(n-1)(\Delta-2) - \log\delta)/\log(\Delta-1)$ for any Δ-critical graph of order n with minimum degree δ. On the other hand, Fiorini [6] showed there exist infinitely many Δ-critical graphs G of order n with $c(G) \leq 3(\Delta+1)2^{q(n)}$, where $q(n) = (\log n - \log(\Delta-1))/\log(2\Delta-2)$. Consequently, there are infinitely many non-Hamiltonian Δ-critical graphs. By increasing maximum degrees in terms the orders of graphs, Luo and Zhao [9] proved that every Δ-critical graph G of order n with $\Delta \geq \frac{6n}{7}$ is Hamiltonian. Furthermore, Luo, Miao and Zhao [11] improved the lower bound to $\Delta \geq \frac{4n}{5}$. It is interesting to determine the minimum positive real number β such that every Δ-critical graph of order n with $\Delta \geq \beta n$ is Hamiltonian. In [9], Luo and Zhao also proved that if G is an overfull Δ-critical graph, then $c(G) \geq \min\{2\Delta, n\}$. So, every Δ-critical overfull graph with $\Delta \geq n/2$ is Hamiltonian. Based on Hilton's overfull graph conjecture [5], we made the following conjecture.

Conjecture 1. *Let G be a Δ-critical graph of order n. If $\Delta > n/3$, then G is Hamiltonian.*

In this paper, we prove the following result.

Theorem 1. *If G is a Δ-critical graph of order n with $\Delta \geq \frac{3n}{4}$, then G is Hamiltonian.*

Our proof is inspired by the recent development of Tashkinov tree technique for graph edge coloring for multigraphs. Kostochka and Stiebitz [13] obtained a nice result for Kierstead paths of order 4. We extend the result to a structure of a broom. We also use a result of Brandt and Veldman's circumference formula in terms of degree sum of two adjacent vertices.

2. Lemmas and notation

The following results about Δ-critical graph are needed in our proof.

Lemma 1 (*Vizing [15]*). *If xy is an edge of a Δ-critical graph G, then x is adjacent to at least $\Delta - d(y) + 1$ vertices z $(z \neq y)$ with degree Δ.*

Lemma 2 (*Luo and Zhao [11]*). *Let G be a Δ-critical graph and d be a positive integer. Then, for any d-vertex x, there does not exist a vertex subset U satisfying the following three conditions.*

1. *$x \notin U$ and $|U| = d - 1$;*
2. *$d(u) \leq \frac{\Delta-d+1}{2}$ for each vertex $u \in U$;*
3. *there are $d-1$ distinct neighbors of x, each of which is adjacent to a distinct vertex in U.*

Lemma 3 (*Luo, Miao and Zhao [9]*). *Let G be a Δ-critical graph and A be an independent set of G. If $A \cap V_\Delta = \emptyset$, then $|N(A)| > |A|$.*

Lemma 4 (*Brandt and Veldman [3]*). *Let $G \neq K_{1,n-1}$ be a graph of order n. If $d(x) + d(y) \geq n$ for any edge xy of G, then there exists an independent set S with $S \cup N(S) \neq V(G)$ such that $c(G) = n - \max\{|S| - |N(S)| + 1, 0\}$.*

Lemma 5. *Let G be a Δ-critical graph of order n with $\Delta \geq \frac{n}{2}$. If $d(x) + d(y) \geq n$ for any edge xy of G, then G is Hamiltonian.*

Proof. Suppose on the contrary that there exists a graph G satisfying the above conditions but is not Hamiltonian. By Lemma 4, G contains an independent set S with $S \cup N(S) \neq V(G)$ such that $n - \max\{|S| - |N(S)| + 1, 0\} \leq n - 1$. Thus $|S| - |N(S)| + 1 \geq 1$. Since $S \cup N(S) \neq V(G)$, S is not a maximum independent set. Since $|S| - |N(S)| + 1 \geq 1$, we have $|S| \geq |N(S)|$. By Lemma 3, S contains a Δ-vertex v. Then $|S| \geq |N(S)| \geq d(v) = \Delta \geq \frac{n}{2}$. It follows that $|S| = |N(S)| = \frac{n}{2}$, and hence S is a maximal independent set, giving a contradiction. □

Let G be a graph with $\chi'(G) \leq k$ and let φ be an edge-k-coloring of G. For any two different colors i and j, an *$i-j$ edge chain* is a component of the graph formed by all edges colored i and j in the coloring φ; it is a path or an even cycle. If we include isolated vertices as $i-j$ edge chains, then for any vertex $v \in V(G)$ there is a unique $i-j$ edge chain $L_{i,j}(v)$ containing v. For any $v \in V(G)$, let $\varphi(v) = \{\varphi(e) : e \in E(G)$ and e is incident to $v\}$, and $\overline{\varphi}(v) = \{1, 2, \ldots, k\} - \varphi(v)$. We call $\varphi(v)$ and $\overline{\varphi}(v)$ the set of *colors present* and the set of *colors missing* at v, respectively. We call a vertex set $S \subseteq V(G)$ *elementary* if for any two distinct vertices $u, v \in S$, $\overline{\varphi}(u) \cap \overline{\varphi}(v) = \emptyset$.

Let G be a graph and $e \in E(G)$ such that $\chi'(G-e) \leq k$, and let $\varphi \in \mathcal{C}^k(G-e)$. A *Kierstead path* with respect to e and φ is defined to be a sequence $K = (y_0, e_1, y_1, \ldots, e_p, y_p)$ with $p \geq 1$ consisting of edges $e_1, \ldots, e_p$ and vertices $y_0, \ldots, y_p$ satisfying the following two conditions:

(1) The vertices $y_0, \ldots, y_p$ are distinct, $e_1 = e$ and $e_i \in E_G(y_i, y_{i-1})$ for $1 \leq i \leq p$;
(2) For every edge e_i with $2 \leq i \leq p$, there is a vertex y_j with $0 \leq j < i$ such that $\varphi(e_i) \in \overline{\varphi}(y_j)$.

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