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A new formula for the decycling number of regular graphs[☆]

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ABSTRACT

The decycling number $\nabla(G)$ of a graph G is the smallest number of vertices which can be removed from G so that the resultant graph contains no cycle. A decycling set containing exactly $\nabla(G)$ vertices of G is called a ∇-set. For any decycling set S of a k-regular graph G, we show that $|S| = \frac{\beta(G)+m(S)}{k-1}$, where $\beta(G)$ is the cycle rank of G, $m(S) = c + |E(S)| - 1$ is the margin number of S, c and $|E(S)|$ are, respectively, the number of components of $G - S$ and the number of edges in $G[S]$. In particular, for any ∇-set S of a 3-regular graph G, we prove that $m(S) = \xi(G)$, where $\xi(G)$ is the Betti deficiency of G. This implies that the decycling number of a 3-regular graph G is $\frac{\beta(G)+\xi(G)}{2}$. Hence $\nabla(G) = \lceil \frac{\beta(G)}{2} \rceil$ for a 3-regular upper-embeddable graph G, which concludes the results in [Gao et al., 2015, Wei and Li, 2013] and solves two open problems posed by Bau and Beineke (2002). Considering an algorithm by Furst et al., (1988), there exists a polynomial time algorithm to compute $Z(G)$, the cardinality of a maximum nonseparating independent set in a 3-regular graph G, which solves an open problem raised by Speckenmeyer (1988). As for a 4-regular graph G, we show that for any ∇-set S of G, there exists a spanning tree T of G such that the elements of S are simply the leaves of T with at most two exceptions providing $\nabla(G) = \lceil \frac{\beta(G)}{3} \rceil$. On the other hand, if G is a loopless graph on n vertices with maximum degree at most 4, then

$$\nabla(G) \le \begin{cases} \dfrac{n+1}{2}, & \text{if } G \text{ is 4-regular,} \\ \dfrac{n}{2}, & \text{otherwise.} \end{cases}$$

The above two upper bounds are tight, and this makes an extension of a result due to Punnim (2006).

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1. Introduction

Let $G = (V(G), E(G))$ be a graph. Graphs considered in this paper are loopless finite, connected and multiple edges are permitted. For general theoretic notations, we follow [5]. The minimum number of edges whose removal eliminates all cycles in a given graph has been known as the *cycle rank* of the graph, and this parameter has a simple expression: $\beta(G) = |E(G)| - |V(G)| + w$ (see [10]), where w is the number of components of G. The corresponding problem of eliminating all cycles from a graph by means of deletion of vertices goes back at least to the work of Kirchhoff on spanning trees in [11].

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A vertex set $S \subseteq V$ is called a *decycling set* of G if $G - S$ is acyclic. The cardinality of a minimum decycling set of G is denoted by $\nabla(G)$ (or ∇ for short). A decycling set contains exactly $\nabla(G)$ vertices of G is called a ∇-set. Vertices of a decycling set are labeled by " $\bullet$ " in the following figures. Let $m(S) = c + |E(S)| - 1$ be the *margin number* of a decycling set S which measures the gap between S and a ∇-set of G, where c and $|E(S)|$ are, respectively, the number of components of $G - S$ and the number of edges in $G[S]$. The problem of determining the decycling number of an arbitrary graph is NP-complete (see [12]). In fact, computing decycling numbers of the following families of graphs are shown to be NP-hard: planar graphs, bipartite graphs and perfect graphs. One may see [2] as a brief survey.

For any two graphs G and H, their *Cartesian product* $G \times H$ is defined as: $V(G \times H) = \{(u_i, v_j)|i = 1, 2, \dots, m, j = 1, 2, \dots, n\}$ and $E(G \times H) = \{(u_i, v_j)(u_r, v_s)|i = r, v_jv_s \in E(H) \text{ or } j = s, u_iu_r \in E(G)\}$, where $V(G) = \{u_1, u_2, \dots, u_m\}$ and $V(H) = \{v_1, v_2, \dots, v_n\}$. $\Delta(G)$ (or Δ for short) represents the maximum degree of a graph G. Let $d_G(x)$ and $N_E(x)$ (or $N_E(S)$) be, respectively, the degree of vertex x and the set of edges incident to vertex x (or the vertex in S, and the edges between vertices in S also belong to $N_E(S)$) of G.

In this paper, we consider the problem of the decycling number from a new perspective: the effects of graph embeddings on the decycling number of graphs. Given a connected graph G and a surface P, we say that G can be *embedded* into P if there exists a polyhedron $\sum$ on P such that the 1-skeleton of $\sum$ has a subgraph homeomorphic to G. The components of $\sum - G$ are called the *faces* of the embedding. When each face is homeomorphic to an open disc, the embedding is called a *cellular*. The maximum genus of a connected graph G, denoted by $\gamma_M(G)$, is the largest genus of an orientable surface on which G admits a cellular embedding. Let T be a spanning tree of a connected graph G. The subgraph $G - E(T)$ of G is called a *co-tree* of G. Note that the number of edges in any co-tree of G is just the cycle rank $\beta(G)$. The *Betti deficiency* of G, denoted by $\xi(G)$, is the minimum number of odd components (i.e., the components containing odd number of edges) among co-trees of G. Any spanning tree whose co-tree achieves the Betti deficiency $\xi(G)$ is called a *Xuong-tree*, and denoted by T_X. A graph G is *upper-embeddable* if and only if $\xi(G) \leq 1$ [13,23].

The maximum genus of a graph can be characterized as follows:

Lemma 1.1 (*[23]*). *Let G be a connected graph. Then*

$$\gamma_M(G) = \frac{\beta(G) - \xi(G)}{2}.$$

Lemma 1.2 (*[19]*). *Let G' be a subdivision of a connected graph G. Then $\nabla(G') = \nabla(G)$.*

Xuong defined an edge-partition of a co-tree in [23].

Lemma 1.3 (*[23]*). *Let G be a connected graph and T_X a Xuong-tree of G. Then there exists an edge-partition of $E(G) - E(T_X)$ as follows:*

$$E(G) - E(T_X) = \{e_1, e_2\} \cup \{e_3, e_4\} \cup \cdots \cup \{e_{2m-1}, e_{2m}\} \cup \{f_1, f_2, \dots, f_s\},$$

where (i) $m = \gamma_M(G)$, $s = \xi(G)$; (ii) *for any* $i = 1, 2, \dots, m$, $e_{2i-1} \cap e_{2i} \neq \emptyset$, *and* $\{f_1, f_2, \dots, f_s\}$ *is a matching of* G.

Let T_X be a Xuong-tree and the edge-partition of $E(G) - E(T_X)$ be defined as Lemma 1.3. Consider a set

$$S_X = \{u_i | u_i \in e_{2i-1} \cap e_{2i}, 1 \leq i \leq m\} \cup \{v_j | v_j \text{ is an end of } f_j, 1 \leq j \leq s\}.$$

Then $G - S_X$ contains no cycle (since removing S_X from G will eliminate all the possible fundamental cycles of T_X) and hence S_X is a decycling set of G, that is, $\nabla(G) \leq |S_X|$.

Corollary 1.1. $\nabla(G) \leq |S_X| \leq \gamma_M(G) + \xi(G)$ *holds for any graph G.*

These are new bounds for the decycling number $\nabla(G)$ of a graph G. In some cases, dense graphs for example, the bounds cannot work well since the values of $|S_X|$ and $\gamma_M(G)$ may be too big. It is clear that the bound $|S_X|$ heavily depends on the choice of Xuong-tree T_X since different T_X may lead to quite different value of $|S_X|$. For instance, the wheel graph $W_{1,n} = K_1 \vee C_n$ with n spokes has $\nabla(W_{1,n}) = 2$. If one chooses a Xuong-tree $K_{1,n}$ as a spanning tree of $W_{1,n}$, then the corresponding $|S_X|$ equals to $\lceil \frac{n}{2} \rceil$; meanwhile, a Hamilton path in $W_{1,n}$ will determine another S_X whose number of elements reaches the best value $\nabla(W_{1,n}) = 2$. Therefore, how to find a set $S_X \subseteq V(G)$ with the smallest size is a key to determine $\nabla(G)$.

The paper is organized as follows.

In Section 2, we prove that $|S| = \frac{\beta(G)+m(S)}{k-1}$ for any decycling set S of a k-regular graph G, which implies that S is a ∇-set if and only if $m(S)$ is minimum. This formula, although contains an uncertain parameter $m(S)$, can be used to locate the lower bounds of the decycling number for regular graphs. Many examples show that the lower bounds may be tight, see [4,9,16,17,19,21,22]. Our result shows that $\nabla(C_m \times C_n) = \frac{mn+m(S)+1}{3}$ for a ∇-set S of $C_m \times C_n$, which equals to Pike's result $\nabla(C_m \times C_n) = \lceil \frac{mn+2}{3} \rceil$ $(m, n \neq 4)$ when $m(S) \leq 1$ (see [15]). Therefore, this provides a way to locate the exact value of $\nabla(G)$ (to find a decycling set S with the minimum $m(S)$). In addition, this formula also implies that for some (4-regular) graphs G of order n, the margin number $m(S)$ may be a linear function on n (i.e., $m(S)$ tends to infinity as $n \to \infty$). For instance, a toroidal 4-regular graph G containing n disjoint $K_5 - e$'s (see Fig. 3) whose decycling number is $2n+1$ and its margin number $m(S) = n + 2$ for a ∇-set S .