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On palindromes with three or four letters associated to the Markoff spectrum

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ABSTRACT

Let $\mathcal{N}^{(2)}$ and $\mathcal{N}^{(5)}$ be the sets of integer solutions of $2x^2+y_1^2+y_2^2 = 4xy_1y_2$ and $5x^2+y_1^2+y_2^2 = 5xy_1y_2$, respectively. The elements of these sets give sequences in the non-discrete part of the Markoff spectrum consisting of the normalized values of arithmetic minima of indefinite quadratic forms. Each of these sets can be represented by a bipartite graph. Using this we can construct quadratic forms giving the values of the sequences. We show that, for a quadratic form f which gives a value of the Markoff spectrum corresponding to an integer solution x of $\mathcal{N}^{(2)}$ and $\mathcal{N}^{(5)}$, the roots of $f(\xi, 1) = 0$ have a periodic continued fraction expansion and the period is a palindrome with fixed prefix and suffix on $\{1, 2, 3\}$ and $\{1, 2, 3, 4\}$, respectively.

1. Introduction

Let $f(\xi, \eta) = a\xi^2 + b\xi\eta + c\eta^2$ be an indefinite quadratic form with real coefficients and with *discriminant* $D(f) = b^2 - 4ac$. The *Markoff spectrum* (for $\mathbb{Q}$) is defined as the set $\mathcal{M} = \left\{\sqrt{D(f)}/\min(f)\right\}$ for all f with $(a, b, c) \in \mathbb{R}^3$ and $D(f) > 0$, where $\min(f) = \inf_{(\xi,\eta)\in\mathbb{Z}^2-\{(0,0)\}} |f(\xi, \eta)|$. The set $\mathcal{M}$ contains ∞ for the case in which $\min(f) = 0$. The discrete part of $\mathcal{M}$ is in $[\sqrt{5}, 3)$ and is described by using the set of integer solutions $\mathcal{K} = \{1, 2, 5, 13, \ldots\}$ of the equation $x^2 + y^2 + z^2 = 3xyz$: $\mathcal{M} \cap [0, 3) = \left\{\sqrt{9 - 4/k^2} \;\; k \in \mathcal{K}\right\}$. For this result we refer the reader to [6,9,11].

Let us recall the Markoff spectrum on a sublattice (see §5 of [11]). Let p be a positive integer. Consider the lattice $\Omega = \mathbb{Z}^2$ of integer points and define a sublattice of index p as $\Omega_p = \{(\xi, \eta) \in \mathbb{Z}^2 \mid \xi \equiv 0 \pmod{p}\}$. We also consider the set $\mathcal{F}_p$ of real binary indefinite quadratic forms f satisfying $D(f) > 0$ and the condition

$$|f(\xi, \eta)| \geq p' \min(f) \quad \text{if } (\xi, \eta) \in \Omega_{p'} - \{(0, 0)\}$$

for all $p'|p$, where p' are positive integers, not necessarily prime numbers. The set $\mathcal{M}^{(p)} = \left\{\sqrt{D(f)}/\min(f)\right\}$ for all $f \in \mathcal{F}_p$ is called the *Markoff spectrum on the sublattice of index* p. From the definition we immediately have $\mathcal{M}^{(p)} \subset \mathcal{M}^{(1)} = \mathcal{M}$.

In this paper we focus on the Markoff spectra on the sublattices of indices 2 and 5. The discrete part of $\mathcal{M}^{(2)}$ is described by using the integer solutions of the equation $2x^2 + y_1^2 + y_2^2 = 4xy_1y_2$. Let $\mathcal{N}^{(2)}(\Lambda) = \{1, 5, 29, 65, \ldots\}$ denote the set of solutions x and let $\mathcal{N}^{(2)}(M) = \{1, 3, 11, 17, \ldots\}$ denote that of y_1 and y_2. The elements of these sets give the discrete part of

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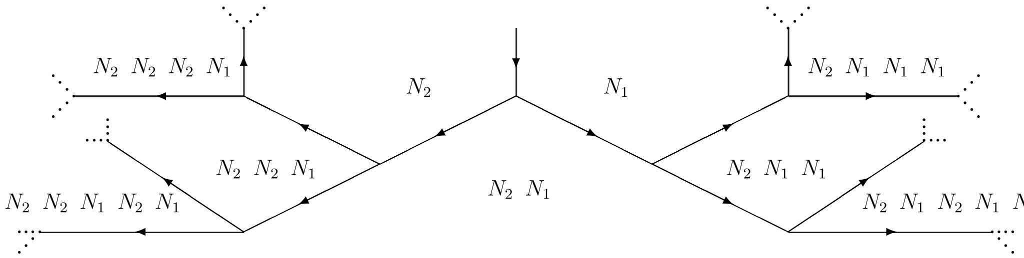

Fig. 1. How to define CM matrices.

$\mathcal{M}^{(2)}$ (see [11,16]):

$$\mathcal{M}^{(2)} \cap [0,4) = \left\{ \sqrt{4^2 - \frac{4}{\lambda^2}} \ \ \lambda \in \mathcal{N}^{(2)}(\Lambda) \right\} \cup \left\{ \sqrt{4^2 - \frac{2\times 4}{m^2}} \ \ m \in \mathcal{N}^{(2)}(M) \right\}. \tag{1.1}$$

The discrete part of the Markoff spectrum $\mathcal{M}^{(5)}$ on the sublattice of index 5 is described by using the integer solutions of the equation $5x^2+y_1^2+y_2^2 = 5xy_1y_2$. Let $\mathcal{N}^{(5)}(\Lambda) = \{1, 2, 5, 13, \ldots\}$ denote the set of solutions x and let $\mathcal{N}^{(5)}(M) = \{1, 2, 3, 7, \ldots\}$ denote that of y_1 and y_2. The elements of these sets give the discrete part of $\mathcal{M}^{(5)}$ (see [11,16]):

$$\mathcal{M}^{(5)} \cap [0,5) = \left\{ \sqrt{5^2 - \frac{4}{\lambda^2}} \ \ \lambda \in \mathcal{N}^{(5)}(\Lambda) \right\} \cup \left\{ \sqrt{5^2 - \frac{5\times 4}{m^2}} \ \ m \in \mathcal{N}^{(5)}(M) \right\}. \tag{1.2}$$

Note that, except for some values, the sequences (1.1) and (1.2) are in the non-discrete part of the Markoff spectrum $\mathcal{M}$.

For each $\alpha \in \mathcal{M} \cap [0, 3)$, we can use an integer solution (x, y, z) of the equation $x^2 + y^2 + z^2 = 3xyz$ to construct a quadratic form (called *Markoff minimal form*) satisfying $\sqrt{D(f)}/\min(f) = \alpha$ (see Theorem 6 in Chapter 1 of [9]). Moreover, it is known that the continued fraction expansion of the roots of $f(\xi, 1) = 0$ has a period with high symmetry (see Chapters 1, 2 of [9] and [13]). It is natural to ask how we define minimal forms which give the values of the discrete part of $\mathcal{M}^{(2)}$ and $\mathcal{M}^{(5)}$, and if a minimal form f is defined, which property the continued fraction expansion of the roots of $f(\xi, 1) = 0$ has.

Here we focus on palindromes in the periods of the continued fraction expansions. Setting $z = 0$ in the right side of (1.4) (see below), we consider a continued fraction. Supposing that a_i, $i \in \mathbb{N}$ are positive integers, we regard the sequence of the partial quotients $a_0a_1 \ldots a_n$ as a word. Indeed, in this paper a_i is 1, 2, 3, or 4. The only exception is 0 occurring in 210. For a given word $w = a_0 \ldots a_n$ (not empty), let w^* denote the reverse $a_n \ldots a_0$ of w. The word w is said to be a *palindrome* if $w = w^*$.

The Markoff spectra on the sublattices of indices 2 and 5 are an object of study from the following geometric point of view (see [3]). Two-generator Fuchsian groups, the quotient space of which is a once punctured torus, are called *Fricke groups* and are parametrized by the equation $X^2 + Y^2 + Z^2 = XYZ$. There are only four arithmetic Fricke groups which are Fricke groups commensurable to a conjugate of the modular group $\mathrm{PSL}(2, \mathbb{Z})$. The equations $x^2 + y^2 + z^2 = 3xyz$, $2x^2 + y_1^2 + y_2^2 = 4xy_1y_2$, and $5x^2 + y_1^2 + y_2^2 = 5xy_1y_2$ are obtained from the three arithmetic Fricke groups $(3, 3, 3)$, $(2\sqrt{2}, 2\sqrt{2}, 4)$, and $(\sqrt{5}, 2\sqrt{5}, 5)$, respectively. The structure of the integer solutions of these equations is represented by using graphs and the graphs provide elements of $\mathrm{SL}(2, \mathbb{R})$ whose $(2, 1)$-entries are the solutions. Taking the fixed point equation of the linear fractional transformation of these matrices, we obtain Markoff minimal forms. To represent two shapes of the solutions of the latter two equations, we need a graph with a bipartite coloring. Let μ be an element of $\mathcal{K}$, $\mathcal{N}^{(2)}(\Lambda)$, or $\mathcal{N}^{(5)}(\Lambda)$. Because of the arithmeticity of the groups, we can take an element M_μ of $\mathrm{SL}(2, \mathbb{Z})$ whose $(2, 1)$-entry is μ. Expanding M_μ in the sense of (1.4), we get a palindrome.

Palindromic periods of the continued fraction expansions are useful for computing sums and an invariant which appear in the analytic number theory. For an element $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $\mathrm{SL}(2, \mathbb{Z})$ with $c > 0$, the Dedekind sum of M is defined by $s(d, c)$ and the Rademacher invariant is defined by $\Psi(M) = (a + d)/c - 12s(d, c) - 3$ (here we use the notation of [12]). If a period of the expansion of M has a palindrome, we easily compute $s(d, c)$ and $\Psi(M)$. Thus, we get the Dedekind sum and the Rademacher invariant of M_μ in the previous paragraph. They characterize the Markoff minimal form defined by M_μ (see [4]).

Palindromes associated to the discrete part of $\mathcal{M}$. H. Cohn linked in [7] a Markoff minimal form to an element of a subgroup $\langle A, B\rangle$ of $\mathrm{SL}(2, \mathbb{Z})$ generated by $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$, and represented in [8] the element as a primitive word of the two-generator free group $\langle A, B\rangle$. A word W is *primitive* in a two-generator free group G if there exists a word V such that W and V generate G. Following Cohn's idea, we will briefly explain a way of constructing Markoff minimal forms. The graph method below can be applied to the construction of minimal forms giving the values of the discrete part of $\mathcal{M}^{(2)}$ and $\mathcal{M}^{(5)}$ (see Section 2 and [3]).

Consider a binary tree properly embedded in the plane as in Fig. 1, where $N_1 = B^{-1}$ and $N_2 = B^{-1}A^{-1}B^{-1}$ are elements of $\langle A, B\rangle$. Suppose that every edge of the tree is directed downward. Following the direction, we successively define new