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On strong graph bundles[☆]

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ABSTRACT

We study *strong graph bundles*: a concept imported from topology which generalizes both covering graphs and product graphs. Roughly speaking, a strong graph bundle always involves three graphs E, B and F and a projection $p : E \to B$ with fiber F (i.e. $p^{-1}(x) \cong F$ for all $x \in V(B)$) such that the preimage of any edge xy of B is trivial (i.e. $p^{-1}(xy) \cong K_2 \boxtimes F$). Here we develop a framework to study which subgraphs S of B have trivial preimages (i.e. $p^{-1}(S) \cong S \boxtimes F$) and this allows us to compare and classify several variations of the concept of strong graph bundle. As an application, we show that the clique operator preserves *triangular graph bundles* (strong graph bundles where preimages of triangles are trivial) thus yielding a new technique for the study of clique divergence of graphs.

1. Introduction

In topology a *fiber bundle* is a space which is locally a product of spaces [41]. This concept has proved to be very important in many fields of mathematics including algebraic geometry, differential geometry and differential topology. Also, fiber bundles play a central role in general relativity. Thus, the importance of fiber bundles in mathematics and physics is difficult to overstate. The analogues of fiber bundles in graph theory, i.e. *graph bundles*, were introduced (as reported in [34]) by Pisanski and Vrabec in a 1982 unpublished preprint, and appeared for the first time (with Shawe-Taylor as an additional author) in [33]. Since there are several notions of a product in graph theory, there are also several notions of a graph bundle. Most works on graph bundles focus on *Cartesian graph bundles* [1–3,5,7,8,12,13,15–23,30,33–35,40,43–47], where graphs are locally a Cartesian product of graphs, but there is also research on *strong graph bundles* [18,30,43,47], *tensor graph bundles* [14,18] and *lexicographic graph bundles* [30]. The one just given is an exhaustive classification of all the papers on graph bundles that we could find. Here we shall focus on strong graph bundles, as the strong product $\boxtimes$ suits our purposes best.

More specifically, a strong graph bundle always involves three graphs E, B, and F and a projection $p : E \to B$. Saying that E is "locally a product" means then that the preimage $p^{-1}(x)$ of each vertex $x \in V(B)$ can be seen as $\{x\} \boxtimes F$ in such a way that the restriction of p is just the first projection, and also the preimage $p^{-1}(xy)$ of each edge $xy \in E(B)$ is isomorphic to $\{xy\} \boxtimes F$, again in a way equally compatible with the first projection. Mohar, Pisanski and Škoviera remarked in [30] that a more natural equivalent definition is obtained by asking that the preimage $p^{-1}(\mathrm{St}(x))$ of the star of each vertex $x \in V(B)$ can

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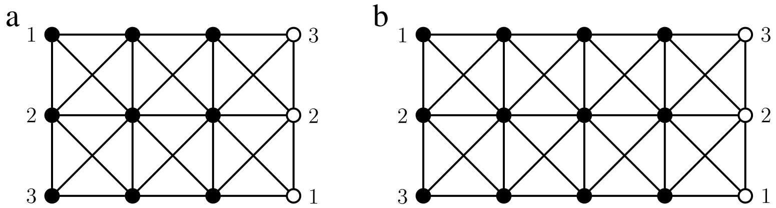

Fig. 1. Two examples. (Identify vertices with equal labels in each graph). (a) A non-triangular graph bundle with base C_3 and fiber P_3. (b) A triangular graph bundle with base C_4 and fiber P_3 where $B = C_4$ is not (p, F)-trivial.

be seen as $\mathrm{St}(x) \boxtimes F$ in such a way that the restriction of p is just the first projection. As far as we know, this is the only notion of locality employed so far in the literature of graph bundles.

But just as the concept of a product of graphs is not unique, neither is so that of locality in a graph, and hence each kind of locality may produce a variant version of graph bundles. We shall study strong graph bundles where these local subgraphs are indeed vertices and edges (or stars), but then we shall explore other types of locality by adding triangles, cliques and closed neighborhoods. It will turn out that all three of them are equivalent (Theorem 3.2), but not equivalent to the original one involving only vertices and edges (Fig. 1a). The new kind of strong graph bundle introduced here will be called *triangular graph bundle*.

Our Theorem 2.8 (together with Lemma 2.1) provides a framework in which different versions of locality for strong graph bundles can be studied and compared. In Theorem 3.2 this result proves the equivalence of our three definitions of a triangular graph bundle, and in Corollary 2.9 it also yields the above-mentioned equivalence [30] of the two definitions of the original strong graph bundles. An interesting and useful tool behind Theorem 2.8 is the concept of *agreement* of graph morphisms at a vertex in Definition 2.3.

We also give an application to clique graphs: Theorem 4.1 states that triangular graph bundles are preserved by the clique operator. This yields, in Theorem 4.2, a new method for proving clique divergence or clique convergence which generalizes and unifies previously known results about strong products [24,31] and triangular covering maps [25].

Let us quickly review the basic terminology now.

Our graphs are simple and finite. The vertex and edge sets of a graph G are denoted by $V(G)$ and $E(G)$, and $|G| = |V(G)|$ is the *order* of G. A graph H is a *subgraph* of G (denoted by $H \leq G$) if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. An *induced subgraph* of G is a subgraph H of G such that whenever $x, y \in V(H)$ and $xy \in E(G)$, we also have $xy \in E(H)$. The *union* of the graphs G and H is given by $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$, and their *intersection* is given by $V(G \cap H) = V(G) \cap V(H)$ and $E(G \cap H) = E(G) \cap E(H)$. Two vertices x, y are *adjacent-or-equal* in G (denoted by $x \simeq y$), if $x = y$ or $xy \in E(G)$. The *closed neighborhood* of $x \in V(G)$ is the subgraph $N_G[x] \leq G$ induced by $\{y \in V(G) \mid x \simeq y\}$.

A *morphism* (or *map*) $f : G \to H$ is a function on the vertex sets $f : V(G) \to V(H)$ such that $x \simeq y \Rightarrow f(x) \simeq f(y)$. In this case we denote the *domain*, *codomain* and *image* of f by $\mathcal{D}_f, \mathcal{C}_f$ and $\mathcal{I}_f$ respectively, i.e. $\mathcal{D}_f = G$, $\mathcal{C}_f = H$, and $\mathcal{I}_f$ is given by $V(\mathcal{I}_f) = \{f(x) \mid x \in V(G)\}$ and $E(\mathcal{I}_f) = \{f(x)f(y) \mid xy \in E(G) \text{ and } f(x) \neq f(y)\}$. Note that $\mathcal{I}_f \leq \mathcal{C}_f$ may be non-induced. Given $f : G \to H$ and $S \leq H$, the *inverse image* of S is the subgraph $f^{-1}(S)$ of G given by $V(f^{-1}(S)) = f^{-1}(V(S))$ and $E(f^{-1}(S)) = \{xy \in E(G) \mid f(x) \simeq f(y) \text{ in } S\}$. Also $f^{-1}(S)$ could be non-induced in G. The *strong product* $G \boxtimes H$ of two graphs is determined by $V(G \boxtimes H) = V(G) \times V(H)$ and $E(G \boxtimes H) = \{(v, w)(v', w') \mid v \simeq v' \text{ in } G \text{ and } w \simeq w' \text{ in } H\}$.

As usual when studying clique graphs, a *complete* of G is a complete subgraph of G, and we reserve the word *clique* for maximal complete subgraphs. The *clique graph* $K(G)$ is the intersection graph of the cliques of G and the operator K is called the *clique operator*. Then the *iterated clique graphs* $K^n(G)$ are defined inductively by $K^0(G) = G$ and $K^n(G) = K(K^{n-1}(G))$. If the sequence $\{|K^n(G)|\}$ is bounded (equivalently, if $K^m(G) \cong K^n(G)$ for some $m > n$), we say that G is *K-convergent*. On the other hand, G is called *K-divergent* if the sequence $\{|K^n(G)|\}$ is unbounded. The *K-behavior* of G can be either K-convergent or K-divergent. This dichotomy is a major topic in the theory of clique graphs, and many papers have appeared providing techniques for determining the K-behavior (e.g. [4,6,9,10,25–29,31,32,36,42]). Applications of the theory of the clique operator include the fixed point property for posets [11] and loop quantum gravity [37–39].

2. Strong graph bundles

Given a graph B, hereinafter referred to as the *base* graph, a *projection* over B is a graph morphism $p : E \to B$ which is vertex-surjective and edge-surjective, i.e. $\mathcal{I}_p = \mathcal{C}_p$. The domain $E = \mathcal{D}_p$ will be called the *total* graph of the projection. The *fiber* of a vertex $x \in V(B)$ is the preimage $p^{-1}(x) \leq E$ of the one-vertex subgraph $\{x\} \leq B$. These *fibers of* p are non-empty induced subgraphs of E. In fact, if $S \leq B$ is induced, then $p^{-1}(S) \leq E$ is induced. Even if S is not induced the restriction of p, denoted also by $p : p^{-1}(S) \to S$, is again a projection.

Any projection $p : E \to B$ partitions $V(E)$ into the disjoint union of the vertex sets of its fibers, so each $v \in V(E)$ lies in a unique fiber of p, namely $v \in V(p^{-1}(x))$ for $x = p(v)$. The projection p being understood, we say that v *lies over* x, or that v *is a vertex over* x. The formula $v = \tilde{x}$ means both that $v \in V(E)$ and $p(v) = x$, so in what follows $\tilde{x}$ shall always denote a vertex of the total graph E lying over the vertex x of the base graph B.