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Some matrix identities on colored Motzkin paths

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ABSTRACT

Merlini and Sprugnoli (2017) give both an algebraic and a combinatorial proof for an identity proposed by Louis Shapiro by using Riordan arrays and a particular model of lattice paths. In this paper, we revisit the identity and emphasize the use of colored partial Motzkin paths as appropriate tool. By using colored Motzkin paths with weight defined according to the height of its last point, we can generalize the identity in several ways. These identities allow us to move from Fibonacci polynomials, Lucas polynomials, and Chebyshev polynomials, to the polynomials of the form $(z+b)^n$.

1. Introduction

In [25], Shapiro introduced a triangle $\left(B_{n,k}\right)_{n,k\geq 0}$, where $B_{n,k}=\frac{k+1}{n+1}\binom{2n+2}{n-k}$. Then the following identity related with this matrix was obtained in Shapiro et al. [27]

$$\sum_{k=0}^{n} B_{n,k}(k+1)=4^n. \tag{1.1}$$

The entries of this matrix have the following combinatorial interpretation (see, for example, [25,36]). Consider a pair of paths that start at the origin, consist of $n+1$ unit steps either east $E=(1,0)$ or north $N=(0,1)$, finishing at the points (a,b) and (c,d). It is assumed that the two paths do not meet after $(0,0)$. Then $B_{n,k}$ counts the number of such pairs of paths for which $c-a=k+1$, i.e., which end a (horizontal) distance $k+1$. Call these partial path pairs. By using this combinatorial interpretation, Woan et al. [36] have given an elegant proof of the above identity. Some other combinatorial interpretations of the identity were given in Callan [5], Cameron and Nkwanta [6], and Chen et al. [7]. Very recently, Merlini and Sprugnoli [18] give both an algebraic and a combinatorial proof for this identity by using Riordan arrays [26] and a particular model of lattice paths, and they also find several generalizations of this identity and obtain a general transformation from an arithmetic into a geometric progression.

In this paper, we will give a combinatorial interpretation and many generalizations of identity (1.1) by using colored partial Motzkin paths and Riordan arrays. A Motzkin path of length n is a lattice path from $(0,0)$ to $(n,0)$ consisting of up steps $U=(1,1)$, horizontal steps $H=(1,0)$ and down steps $D=(1,-1)$ that never goes below the x-axis. A (u,h,d)-colored Motzkin path is a Motzkin path such that the up steps, horizontal steps and down steps are labeled by u colors, h colors and d colors, respectively. In the literatures, the $(1,h,1)$-colored Motzkin paths are called the h-colored Motzkin paths, while

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the $(1, h, d)$-colored Motzkin paths are called the (h, d)-colored Motzkin paths [7,12,24,35]. It is well known that the set of Motzkin paths of length n is enumerated by the Motzkin numbers M_n with generating function $M(t) = \frac{1-t-\sqrt{1-2t-3t^2}}{2t^2}$, and the set of 2-colored Motzkin paths of length n is counted by the Catalan numbers C_{n+1}. A (u, h, d)-colored partial Motzkin path, also called a (u, h, d)-colored Motzkin path ending an (n, k), is defined as an initial segment of a (u, h, d)-colored Motzkin path with terminal point (n, k). Let $\mathcal{M}_{n,k}$ denote the set of all 2-colored Motzkin paths ending at (n, k), where $\mathcal{M}_{0,0} = \{\varepsilon\}$ and ε is the empty path. Cameron and Nkwanta [6] showed that the (n, k)th entry of Shapiro's matrix in identity (1.1) is $|\mathcal{M}_{n,k}|$, and they presented a combinatorial proof of this identity. Chen et al. [7] find many extensions of identity (1.1) by using colored Motzkin paths.

In the next section, by using the method of Merlini and Sprugnoli [19] and the 2-colored Motzkin paths, we obtain a combinatorial proof of identity (1.1). In addition, we establish a bijection ϕ between the set $\mathcal{B}_{n,k}$ of partial path pairs of length $n+1$ and distance $k+1$ and the set $\mathcal{M}_{n,k}$ of 2-colored Motzkin paths ending at (n, k). Then by considering $(1, b, c)$-colored Motzkin paths, we are able to get identities involving the Fibonacci polynomials and the sequence $(1, z+b, (z+b)^2, \ldots)$. In Section 3, we consider three kinds of colored Motzkin paths with privileged steps on the x-axis, and we obtain further identities involving the Lucas polynomials and Chebyshev polynomials.

At the end of this section, we briefly recall the notion of Riordan arrays [8,14,26,29]. An infinite lower triangular matrix $G = (g_{n,k})_{n,k\in\mathbb{N}}$ is called a Riordan array if its column k has generating function $d(t)h(t)^k$, where $d(t) = \sum_{n=0}^{\infty} d_n t^n$ and $h(t) = \sum_{n=1}^{\infty} h_n t^n$ are formal power series with $d_0 \neq 0$ and $h_1 \neq 0$. The Riordan array corresponding to the pair $d(t)$ and $h(t)$ is denoted by $(d(t), h(t))$, and its generic entry is $g_{n,k} = [t^n]d(t)h(t)^k$, where $[t^n]$ denotes the coefficient operator.

The set of all proper Riordan arrays forms a group under ordinary row-by-column product with the multiplication identity $(1, t)$. The product of two Riordan arrays is given by

$$(d(t), h(t))(g(t), f(t)) = (d(t)g(h(t)), f(h(t))), \tag{1.2}$$

and the inverse of $(d(t), h(t))$ is the Riordan array

$$(d(t), h(t))^{-1} = (1/d(\bar{h}(t)), \bar{h}(t)), \tag{1.3}$$

where $\bar{h}(t)$ is compositional inverse of $h(t)$, i.e., $h(\bar{h}(t)) = \bar{h}(h(t)) = t$.

If $(b_n)_{b\in\mathbb{N}}$ is any sequence having $b(t) = \sum_{n=0}^{\infty} b_n t^n$ as its generating function, then for every Riordan array $(d(t), h(t)) = (g_{n,k})_{n,k\in\mathbb{N}}$

$$\sum_{k=0}^{n} g_{n,k} b_k = [t^n]d(t)b(h(t)). \tag{1.4}$$

This is called the fundamental theorem of Riordan arrays and it can be rewritten as

$$(d(t), h(t))b(t) = d(t)b(h(t)). \tag{1.5}$$

A Riordan array $G = (d(t), h(t)) = (g_{n,k})_{n,k\in\mathbb{N}}$ can be characterized [14,16,23,29] by two sequences, the A-sequence, $A = (a_n)_{n\in\mathbb{N}}$ and, the Z-sequence, $Z = (z_n)_{n\in\mathbb{N}}$ such that

$$\begin{aligned} g_{n+1,0} &= z_0 g_{n,0} + z_1 g_{n,1} + z_2 g_{n,2} + \cdots + z_n g_{n,n}, \\ g_{n+1,k+1} &= a_0 g_{n,k} + a_1 g_{n,k+1} + a_2 g_{n,k+2} + \cdots + a_{n-k} g_{n,n}, \end{aligned}$$

for all $n, k \geq 0$. If $A(t)$ and Z(t) are the generating functions for the corresponding A- and Z-sequences, respectively, then it follows that

$$d(t) = \frac{1}{1 - tZ(h(t))}, \quad \text{and} \quad h(t) = tA(h(t)). \tag{1.6}$$

Furthermore, if the inverse of $(d(t), h(t))$ is $(d(t), h(t))^{-1} = (g(t), f(t))$, then we have

$$f(t) = \frac{t}{A(t)}, \quad \text{and} \quad g(t) = 1 - \frac{tZ(t)}{A(t)}. \tag{1.7}$$

For example, the Shapiro's array in identity (1.1) corresponds to the Riordan array $B = (C(t)^2, tC(t)^2)$, where $C(t) = \frac{1-\sqrt{1-4t}}{2t}$ is the generating function for the Catalan numbers. By the properties of Riordan arrays, the identity (1.1) can be rewritten as

$$\left(C(t)^2, tC(t)^2\right) \frac{1}{(1-t)^2} = \frac{1}{1-4t}. \tag{1.8}$$

By using Riordan arrays and a particular model of lattice paths, Merlini and Sprugnoli [19] have given both an algebraic and a combinatorial proof of the following identity (see also [11])

$$\left(C(t), tC(t)^2\right) \frac{1+t}{(1-t)^2} = \frac{1}{1-4t}. \tag{1.9}$$

We will present a combinatorial interpretation of the matrix identity (1.8) by using the 2-colored Motzkin paths.