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Note

Edge-partitions of graphs and their neighbor-distinguishing index

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ABSTRACT

A proper edge coloring is neighbor-distinguishing if any two adjacent vertices have distinct sets consisting of colors of their incident edges. The minimum number of colors needed for a neighbor-distinguishing edge coloring is the neighbor-distinguishing index, denoted by $\chi'_a(G)$. A graph is normal if it contains no isolated edges. Let G be a normal graph, and let $\Delta(G)$ and $\chi'(G)$ denote the maximum degree and the chromatic index of G, respectively. We modify the previously known techniques of edge-partitioning to prove that $\chi'_a(G) \leq 2\chi'(G)$, which implies that $\chi'_a(G) \leq 2\Delta(G)+2$. This improves the result in Wang et al. (2015), which states that $\chi'_a(G) \leq \frac{5}{2}\Delta(G)$ for any normal graph. We also prove that $\chi'_a(G) \leq 2\Delta(G)$ when $\Delta(G) = 2^k$, k is an integer with $k \geq 2$.

1. Introduction

All graphs considered in this paper are simple and finite. Let $V(G)$, $E(G)$, and $\Delta(G)$ denote the vertex set, the edge set, and the maximum degree of a graph G, respectively. Let $N_G(v)$ and $\deg_G(v) = |N_G(v)|$ denote the set of neighbors and the degree of a vertex v in G, respectively. An *edge-partition* of a graph G into subgraphs $G_1, \ldots, G_m$ is a decomposition of G such that $E(G) = \bigcup_{i=1}^{m} E(G_i)$ and $E(G_i) \cap E(G_j) = \emptyset$ for any pair $i \neq j$. For a graph G and any $S \subseteq E(G)$, the *edge-induced subgraph* $G[S]$ is the subgraph of G whose edge set is S and whose vertex set consists of all end vertices of the edges in S.

A proper edge *k-coloring* of a graph G is a function $\phi : E(G) \to \{1, \ldots, k\}$ such that every two adjacent edges receive different colors. The *chromatic index* $\chi'(G)$ of a graph G is the minimum positive integer k for which G has a proper edge k-coloring. Given an edge k-coloring ϕ of G, we use $C_\phi(v)$ to denote the set of colors assigned to the edges incident with v. The edge coloring ϕ is called *neighbor-distinguishing* (in some papers *adjacent vertex distinguishing*), or nde-coloring for short, if $C_\phi(u) \neq C_\phi(v)$ for any pair of adjacent vertices u and v.

The neighbor-distinguishing index $\chi'_a(G)$ of a graph G is the smallest integer k such that G has a k-nde-coloring. A graph G is *normal* if it contains no isolated edges. It is obvious that G has an nde-coloring if and only if G is normal; thus we consider only normal graphs when examining an nde-coloring.

Zhang, Liu and Wang [7] introduced and investigated a neighbor-distinguishing edge coloring of graphs, where they proposed the following conjecture.

Conjecture 1. *If G is a connected normal graph different from a 5-cycle, then* $\chi'_a(G) \leq \Delta(G) + 2$.

Akbari, Bidkhori and Nosrati [1] proved that $\chi'_a(G) \leq 3\Delta(G)$ for any normal graph G. Zhang, Wang and Lih [6] proved a better upper bound, $\chi'_a(G) \leq \frac{5}{2}\Delta(G)+5$. Wang, Wang and Huo [5] improved this result by showing that $\chi'_a(G) \leq \frac{5}{2}\Delta(G)$. Our main result is the following improvement of the upper bound for χ'_a.

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Theorem 2. *If G is a normal graph, then $\chi'_a(G) \leq 2\chi'(G)$.*

According to the famous Vizing's theorem [4], $\chi'(G) \leq \Delta(G)+1$. Thus as an immediate consequence we have the following corollary.

Corollary 3. *If G is a normal graph, then $\chi'_a(G) \leq 2\Delta(G) + 2$.*

Furthermore, $\chi'_a(G)$ of a normal graph G with $\Delta(G) = 2^k$, $k \in \mathbb{N}$, $k > 2$, does not exceed $2\Delta(G)$, as stated in the next theorem.

Theorem 4. *If G is a normal graph with $\Delta(G) \leq 2^k$, where k is an integer, and $k \geq 2$, then $\chi'_a(G) \leq 2^{k+1}$.*

The proofs of Theorems 2 and 4 are deferred to Section 3. First, we prove the statements that we will use in the proofs of these two theorems.

2. Preliminaries

Ballister et al. [2] proved the first part, while Wang et al. [5] proved the second and third parts of the following theorem.

Theorem 5. *Let G be a normal graph.*

1. *If $\Delta(G) \leq 3$, then $\chi'_a(G) \leq 5$.*
2. *If $\Delta(G) \leq 4$, then $\chi'_a(G) \leq 8$.*
3. *If $\Delta(G) \leq 5$, then $\chi'_a(G) \leq 10$.*

The following lemma and theorem, proved in [6], are the main tools used in papers [6] and [5] to attain the upper bounds.

Lemma 6. *If normal graph G has an edge-partition into normal graphs G_1 and G_2, then $\chi'_a(G) \leq \chi'_a(G_1) + \chi'_a(G_2)$.*

Theorem 7. *Let G be a normal graph with $\Delta(G) \geq 6$. Then there is an edge-partition of G into normal graphs H_1 and H_2, such that:*

1. $\Delta(H_1) \leq 3$,
2. $\Delta(H_2) \leq \Delta(G) - 2$.

The bound $\chi'_a(G) \leq \frac{5}{2}\Delta(G)$ was proved by repeatedly applying the theorem above, and relying on the bounds of $\chi'_a(H)$ for graphs with $\Delta(H) \leq 5$.

We propose a procedure that is also edge-partitioning of a graph into two normal graphs, and then make use of Lemma 6. The difference is that we show how to produce an edge-partition of a normal graph G that has $\chi'(G) = k$ into normal graphs H_1 and H_2 with $\chi'_a(H_1) \leq 4$ and $\chi'(H_2) \leq k - 2$. Consequently, using the bounds from Theorem 5 we get that $\chi'_a(G) \leq 2\chi'(G)$ for any normal graph G.

Define a function $\mathrm{w}_G : E(G) \to \mathbb{N}$ by

$$\mathrm{w}_G(uv) = \deg(u) + \deg(v)$$

for each $uv \in E(G)$. We use the next theorem in the proofs of Theorems 2 and 4 to show that a minimal counterexample of a graph does not contain an edge uv with $\mathrm{w}_G(uv) \leq \Delta(G) + 2$.

Theorem 8. *Assume that there exists a connected graph G with $\Delta(G) \geq 4$ and $\chi'_a(G) > 2k$ where $\Delta(G) \leq k$. If $\mathrm{w}_G(uv) \leq \Delta(G)+2$ for some $uv \in E(G)$, then there exists a normal graph H with $\Delta(H) \leq \Delta(G)$, $|E(H)| < |E(G)|$, and $\chi'_a(H) > 2k$.*

Proof. Suppose that the statement is false. Let uv be an edge with $\mathrm{w}_G(uv) \leq \Delta(G) + 2$, and let $H = G - uv$. Clearly, $\Delta(H) \leq \Delta(G)$ and $|E(H)| < |E(G)|$. Thus if H is a normal graph, then $\chi'_a(H) \leq 2k$. Denote by L the set of colors $\{1, \ldots, 2k\}$.

Assume first that H is not a normal graph. Since G is a normal graph, this means that one of the vertices u and v, say v, has only one adjacent vertex w in H, and $\deg_H(w) = 1$. Now let $H' = G - vw$. Then $\deg_{H'}(v) = 1$ and $\deg_{H'}(w) = 0$. Also, $\deg_{H'}(u) > 1$ since G is a connected graph with $\Delta(G) \geq 4$. Thus H' is a normal graph. Then there exists an nde-coloring σ of H' with colors from L. Let ϕ be an edge coloring of G with $\phi(e) = \sigma(e)$ for every $e \in E(H')$. Assign to vw any color from L different from $\phi(uv)$, and if $\deg_G(u) = \deg_G(v) = 2$, a color not in $C_\phi(u)$. Since $|L| > 2$, this can always be done; thus ϕ is an nde-coloring with not more than $2k$ colors. This produces a contradiction.

Therefore, we may assume that H is a normal graph, implying that $\chi'_a(H) \leq 2k$. Depending on whether vertices u and v have a common neighbor, we consider two cases.

1. $N_G(u) \cap N_G(v) = \emptyset$. Let H' be a graph obtained from G by contracting the edge uv, and denote by w the vertex obtained by identifying u and v. Since $\deg_{H'}(w) = \deg_G(u) + \deg_G(v) - 2 \leq \Delta(G)$, it follows that $\Delta(H') \leq \Delta(G)$. Hence $|E(H')| < |E(G)|$ and so $\chi'_a(H') \leq 2k$. Let σ' be an nde-coloring of H' with colors from L. Next, define an edge coloring σ of H in the following manner:

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