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Hankel determinants of linear combinations of consecutive Catalan-like numbers

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ABSTRACT

Let $(a_n)_{n\geq 0}$ be a sequence of the Catalan-like numbers. We evaluate Hankel determinants $\det[\lambda a_{i+j}+\mu a_{i+j+1}]_{0\leq i,j\leq n}$ and $\det[\lambda a_{i+j+1}+\mu a_{i+j+2}]_{0\leq i,j\leq n}$ for arbitrary coefficients λ and μ. Our results unify many known results of Hankel determinant evaluations for classic combinatorial counting coefficients, including the Catalan, Motzkin and Schröder numbers.

1. Introduction

Given a sequence $(a_n)_{n\geq 0}$, define its *Hankel matrix* $[a_{i+j}]_{i,j\geq 0}$ and the nth *Hankel determinant* $\det[a_{i+j}]_{0\leq i,j\leq n}$. Hankel determinants occur naturally in diverse areas of mathematics. In recent years, there has been a considerable amount of interest in the evaluation of Hankel determinants $\det[a_{i+j+m}]_{0\leq i,j\leq n}$ and $\det[a_{i+j+m}+a_{i+j+m+1}]_{0\leq i,j\leq n}$ involving various combinatorial sequences [1–3,5–12,14–16,18]. As we will see in Examples 2.4 and 2.5, these combinatorial sequences, including the Catalan numbers, the Motzkin numbers and the Schröder numbers, turn out to be the so-called Catalan-like numbers (or generalized Motzkin numbers [19]). The purpose of this paper is to provide a unified framework for previous results from the viewpoint of Catalan-like numbers.

Let $\mathfrak{s}=(s_k)_{k\geq 0}$ and $\mathfrak{t}=(t_k)_{k\geq 1}$ be two sequences of nonnegative numbers and define an infinite lower triangular matrix $A=[a_{n,k}]_{n,k\geq 0}$ by the recurrence

$$a_{0,0}=1,\quad a_{n+1,k}=a_{n,k-1}+s_k a_{n,k}+t_{k+1}a_{n,k+1}, \tag{1.1}$$

where $a_{n,k}=0$ unless $n\geq k\geq 0$. Clearly, all $a_{n,n}=1$. Following Aigner [3], we say that A is the *recursive matrix* and $a_n=a_{n,0}$ are the nth *Catalan-like numbers* corresponding to $(\mathfrak{s},\mathfrak{t})$.

Example 1.1. The Catalan-like numbers unify many well-known counting coefficients, such as

(i) the Catalan numbers C_n when $\mathfrak{s}=(1,2,2,\ldots)$ and $\mathfrak{t}=(1,1,1,\ldots)$;
(ii) the shifted Catalan numbers C_{n+1} when $\mathfrak{s}=(2,2,2,\ldots)$ and $\mathfrak{t}=(1,1,1,\ldots)$;
(iii) the Motzkin numbers M_n when $\mathfrak{s}=\mathfrak{t}=(1,1,1,\ldots)$;
(iv) the central binomial coefficients $\binom{2n}{n}$ when $\mathfrak{s}=(2,2,2,\ldots)$ and $\mathfrak{t}=(2,1,1,\ldots)$;

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(v) the central trinomial coefficients T_n when $\mathfrak{s} = (1, 1, 1, \ldots)$ and $\mathfrak{t} = (2, 1, 1, \ldots)$;
(vi) the central Delannoy numbers D_n when $\mathfrak{s} = (3, 3, 3, \ldots)$ and $\mathfrak{t} = (4, 2, 2, \ldots)$;
(vii) the large Schröder numbers r_n when $\mathfrak{s} = (2, 3, 3, \ldots)$ and $\mathfrak{t} = (2, 2, \ldots)$;
(viii) the little Schröder numbers S_n when $\mathfrak{s} = (1, 3, 3, \ldots)$ and $\mathfrak{t} = (2, 2, \ldots)$;
(ix) the Fine numbers F_n when $\mathfrak{s} = (0, 2, 2, \ldots)$ and $\mathfrak{t} = (1, 1, 1, \ldots)$;
(x) the Riordan numbers R_n when $\mathfrak{s} = (0, 1, 1, \ldots)$ and $\mathfrak{t} = (1, 1, 1, \ldots)$;
(xi) the (restricted) hexagonal numbers h_n when $\mathfrak{s} = (3, 3, 3, \ldots)$ and $\mathfrak{t} = (1, 1, 1, \ldots)$;
(xii) the Bell numbers B_n when $\mathfrak{s} = \mathfrak{t} = (1, 2, 3, 4, \ldots)$;
(xiii) the factorial $n!$ when $\mathfrak{s} = (1, 3, 5, 7, \ldots)$ and $\mathfrak{t} = (1, 4, 9, 16, \ldots)$.

The Catalan-like numbers have a nice combinatorial interpretation from the viewpoint of weighted lattice paths. A Motzkin path of length n is a lattice path from $(0, 0)$ to $(n, 0)$ consisting of up steps $(1, 1)$, down steps $(1, -1)$ and horizontal steps $(1, 0)$ that never falls below the x-axis. The height of a step in a Motzkin path is the y coordinate of the starting point. Assign a weight 1 (s_k, t_k, resp.) to all up steps (all horizontal steps, all down steps, resp.) of height k. Define the weight of a Motzkin path to be the product of weights of its steps. Then the Catalan-like number a_n counts the total weight of all Motzkin paths of length n.

The Catalan-like numbers are closely related to continued fractions and orthogonal polynomials. Let a_n be the Catalan-like numbers corresponding to $(\mathfrak{s}, \mathfrak{t})$. Then

$$\sum_{n\geq 0} a_n x^n = \cfrac{1}{1 - s_0 x - \cfrac{t_1 x^2}{1 - s_1 x - \cfrac{t_2 x^2}{1 - s_2 x - \cdots}}}.$$

Let $(p_n(x))_{n\geq 0}$ be the sequence of orthogonal polynomials with respect to the linear operator $\mathcal{L}(x^n) = a_n$. Then $\mathcal{L}(p_m(x)p_n(x)) = \delta_{m,n} t_1 \cdots t_n$ and

$$p_{n+1}(x) = (x - s_n)p_n(x) - t_n p_{n-1}(x), \quad p_0(x) = 1.$$

For an infinite matrix $M = [m_{i,j}]_{i,j\geq 0}$, let $M_n = [m_{i,j}]_{0\leq i,j\leq n}$ denote its nth leading principal submatrix and $\delta_n(M) = \det M_n$. For convenience, denote $\delta_{-1}(M) = 1$. Let

$$J = \begin{bmatrix} s_0 & 1 & & & \\ t_1 & s_1 & 1 & & \\ & t_2 & s_2 & 1 & \\ & & t_3 & s_3 & \ddots \\ & & & \ddots & \ddots \end{bmatrix}$$

denote the coefficient matrix of the recursive relation (1.1) and $d_n = \delta_n(J)$. Denote $T_0 = 1$ and $T_n = t_1 t_2 \cdots t_n$ for $n \geq 1$. The following result is folklore (see [3,19] for instance).

Proposition 1.2. *Let a_n be the Catalan-like numbers corresponding to $(\mathfrak{s}, \mathfrak{t})$. Then*

(i) $\det[a_{i+j}]_{0\leq i,j\leq n} = T_1 \cdots T_n$.
(ii) $\det[a_{i+j+1}]_{0\leq i,j\leq n} = T_1 \cdots T_n d_n$.
(iii) $\det[a_{i+j+2}]_{0\leq i,j\leq n} = T_1 \cdots T_n T_{n+1} \sum_{i=-1}^{n} \frac{d_i^2}{T_{i+1}}$.

For $\lambda, \mu \in \mathbb{R}$, let

$$d_n^{(\lambda,\mu)} = \det \begin{bmatrix} \lambda + \mu s_0 & \mu & & & \\ \mu t_1 & \lambda + \mu s_1 & \mu & & \\ & \mu t_2 & \ddots & \ddots & \\ & & \ddots & \lambda + \mu s_{n-1} & \mu \\ & & & \mu t_n & \lambda + \mu s_n \end{bmatrix}.$$

Then $d_n^{(1,0)} = 1$ and $d_n^{(0,1)} = d_n$. Our main results are the following general formulae.

Theorem 1.3. *Let a_n be the Catalan-like numbers corresponding to $(\mathfrak{s}, \mathfrak{t})$. Then*

(i) $\det[\lambda a_{i+j} + \mu a_{i+j+1}]_{0\leq i,j\leq n} = T_1 \cdots T_n d_n^{(\lambda,\mu)}$.
(ii) $\det[\lambda a_{i+j+1} + \mu a_{i+j+2}]_{0\leq i,j\leq n} = T_1 \cdots T_n T_{n+1} \sum_{i=-1}^{n} \frac{d_i d_i^{(\lambda,\mu)}}{T_{i+1}} \mu^{n-i}$.

In the next section, we give the proof of the theorem and then present applications on some interesting Catalan-like numbers. Our results unify many known results of Hankel determinant evaluations for classic combinatorial counting coefficients, including the Catalan, Motzkin and Schröder numbers.

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