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The asymmetric VPN tree problem: polyhedral results and Branch-and-Cut

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Abstract

In this paper we consider a variant of the *virtual private network design problem* (VPNDP). Given an uncapacitated physical network, represented by a graph $G = (V \cup P, E)$, where V is the set of VPN routers and P is the set of clients for which it is given thresholds on the amount of traffic that each client can send (b_p^+) or receive (b_p^-), the VPNDP asks for (1) a connected sub-network $G' = (V' \cup P, E')$, (2) a client assignments (p, v), $p \in P$ and $v \in V'$, and (3) a bandwidth allocation u_e, $e \in E'$, in order to accommodate any traffic demand matrix that respects client thresholds. When G' is acyclic, we have a VPN tree (VPNT). Also, when client thresholds are asymmetric, *i.e.*, $\sum_{p\in P} b_p^+ \neq \sum_{b\in P} b_p^-$, the problem has been shown to be NP-hard. In this paper, we give MILP formulations for the asymmetric VPN tree problem. Also, we discuss the polytope associated with one of these formulations and describe several classes of valid inequalities. Moreover, we present necessary and sufficient conditions under which these inequalities define facets. We also devise separation routines. Using these routines, we propose a Branch-and-Cut algorithm and present a computational study.

Keywords: Asymmetric VPN tree, polyhedral approach, facet, Branch-and-Cut.

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1 Introduction

The general VPN Design Problem (VPNDP) can be presented in terms of a graph $G = (V \cup P, E)$, where P is the set of VPN clients and V is the set of network routers. Each edge in E represents a link between either two routers or a router and a client. We can assume, without loss of generality, that every client $p \in P$ is connected to a single router $i \in V$ and that the edges in E have unlimited capacities.

In the very popular *hose workload model* introduced by Duffield *et al.* [1], the bandwidth requirements of VPN clients can be modeled by defining thresholds $b_p^+ \geq 0$ and $b_p^- \geq 0$, $p \in P$, which represent the expected amount of data that client p can send and receive, respectively. Accordingly to this model, a demand traffic matrix $D \in \mathbb{R}_+^{P \times P}$ is said to be feasible if $\sum_{q \in P} D_{pq} \leq b_p^+$ and $\sum_{p \in P} D_{pq} \leq b_q^-$, for every $p, q \in P$, where the matrix entry D_{pq} represents the amount of information that client p can send to client q.

The minimum VPNDP consists in finding a subgraph $G' = (V' \cup P, E')$ of G spanning all the clients in P and a bandwidth allocation u_e, $e \in E'$, capable of routing any feasible demand traffic matrix, such that, the total bandwidth $\sum_{e \in E'} u_e$ is minimum. When G' is acyclic, we have a VPN tree. In this case, the problem is also referred as the minimum VPN Tree Problem (or VPNTP, for short). This problem has a wide range of practical applications, specially for Internet Service Providers [2].

Accordingly to the hose thresholds, one can say that the VPNTP is *symmetric* when $b_p^+ = b_p^-$ for all $p \in P$, and *balanced* when $\sum_{p \in P} b_p^+ = \sum_{p \in P} b_p^-$ (clearly, any symmetric VPNTP is also balanced, but the converse is not always true). In both cases, the problem has been shown to be polynomially solvable by Gupta *et al.* [2] and Italiano *et al.* [3], respectively. However, in its *asymmetric* version, *i.e.*, when $\sum_{p \in P} b_p^+ \neq \sum_{p \in P} b_p^-$, Gupta *et al.* [2] have shown that the Steiner Tree can be reduced to VPNTP, implying that the latter is NP-hard.

In this paper, we are interested in the *Asymmetric VPN Tree Problem* (AVPNTP, for short) from a polyhedral point of view. We propose two different MILP formulations and show that they are equivalent for the AVPNTP. Then, we discuss the polytope associated with one of these formulations, describe several classes of valid inequalities and derive necessary and sufficient conditions under which these inequalities are facet defining. We also discuss separation routines. These results are used afterwards to develop a Branch-and-Cut algorithm along with computational results are presented.

Although the problems related to designing efficient VPN networks have