Available online at www.sciencedirect.com
ScienceDirect
Electronic Notes in DISCRETE MATHEMATICS
Electronic Notes in Discrete Mathematics 63 (2017) 3–10
www.elsevier.com/locate/endm

Forbidden Induced Subgraphs

Thomas Zaslavsky [1]

Department of Mathematical Sciences
Binghamton University (State University of New York)
Binghamton, NY 13902-6000, U.S.A.

Abstract

In descending generality I survey: five partial orderings of graphs, the induced-subgraph ordering, and examples like perfect, threshold, and mock threshold graphs. The emphasis is on how the induced subgraph ordering differs from other popular orderings and leads to different basic questions.

Keywords: Partial ordering of graphs, hereditary class, induced subgraph ordering, perfect graph, mock threshold graph.

1 Preparation

Graphs can be partially ordered in many ways, of which five make an interesting comparison. This is a very small survey of those partial orderings, their hereditary graph classes (mainly in terms of induced subgraphs), and characterizations by forbidden containments and by vertex orderings, leading up to a new graph class and a new theorem.

Our graphs, written $G = (V, E)$, $G' = (V', E')$, etc., will be finite, simple (with some exceptions), and unlabelled; thus we are talking about isomor-

[1] Email: zaslav@math.binghamton.edu

https://doi.org/10.1016/j.endm.2017.10.056

phism types (isomorphism classes) rather than individual labelled graphs. We list some notation of which we make frequent use:

$$N(G;v) = \text{the neighborhood of } v \text{ in } G.$$
$$d(G;v) = \text{the degree of vertex } v \text{ in } G.$$
$$G{:}X = \text{the subgraph of } G \text{ induced by } X \subseteq V.$$

2 Five Kinds of Containment

We say a graph G *contains* H if $H \leq G$, where $\leq$ denotes some partial ordering on graphs. There are many kinds of containment; each yields a different characterization of each interesting hereditary graph class.

Two kinds of graph that make good models are planar and outerplanar graphs. A *planar graph* can be drawn in the plane with no crossing edges. An *outerplanar graph* can be drawn with no crossing edges and with all vertices on the boundary of the infinite region. These two types can be characterized by forbidden contained graphs.

1. The *subgraph* ordering: $G' \subseteq G$ means $V' \subseteq V$ and $E' \subseteq E$. (Technically, this definition applies to labelled graphs. We apply it to unlabelled graphs by forgetting the labels.). Examples:

(i) Planarity. There are infinitely many forbidden subgraphs, i.e., graphs such that G is planar iff it contains none of those graphs.

(ii) Outerplanarity. There are also infinitely many forbidden subgraphs.

But both infinities are repaired by the next form of containment. *Subdividing* a graph means replacing each edge by a path of positive length (thus, G is a subdivision of itself).

2. The *subdivided subgraph* or *topological subgraph* ordering: $G' \subseteq_t G$ means there is a subdivision G'' of of G' such that $G'' \subseteq G$. Examples:

(i) Planarity. By Kuratowski's Theorem there are two forbidden subdivided subgraphs (K_5 and $K_{3,3}$, in case anyone forgot).

(ii) Outerplanarity. There are also two forbidden subdivided subgraphs, K_4 and $K_{2,3}$, by Chartrand and Harary [2].

3. The *minor* ordering: $G' \preceq G$ means G has a subgraph that contracts to G'. (Contraction means shrinking an edge to a point, thereby combining the endpoints into a single vertex. Or, several edges may be contracted.) Examples: