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Negative Circles in Signed Graphs: A Problem Collection

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Abstract

I propose that most problems about circles (cycles, circuits) in ordinary graphs that have odd or even length find their proper setting in the theory of signed graphs, where each edge has a sign, + or −. Even-circle and odd-circle problems correspond to questions about positive and negative circles in signed graphs. (The sign of a circle is the product of its edge signs.) I outline questions about circles in signed graphs, that seem natural and potentially important.

Keywords: Signed graph, negative circle, positive circle, negative cycle, positive cycle, graph decomposition, counting negative circles, frustration index, frustration number

1 Beginning

Problems about circles (cycles, circuits) in ordinary graphs have attracted much attention over the years. Problems about circles of odd length, or even length, form a small but increasing important part of that area. I propose

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that the proper setting for questions involving parity of circles is, mostly, the theory of signed graphs, where each edge has a sign. Even-circle and odd-circle problems for unsigned graphs generalize, respectively, to questions about positive and negative circles in signed graphs. Here I outline a framework for questions about circles in signed graphs that seem natural and potentially valuable. Some of the questions arise from other research; most are simply basic structural questions needed for a better understanding of signed graphs.

A *signed graph* is $\Sigma = (\Gamma, \sigma)$, where Γ is a graph and $\sigma : E(\Gamma) \to \{+, -\}$, the *signature*, is a sign function on the edges. I assume all graphs are simple and I write $n := |V|$, the order of the graph. The *sign of a circle* is the product of the signs of its edges. A crucial property of a signed graph is the list of its positive circles. Some elementary properties that depend only on that list:

Balance: Σ is balanced if all circles are positive. Otherwise it is unbalanced, or (in physics) frustrated. For instance, the all-positive signed graph $(\Gamma, +)$ is balanced.

Balancing edge: Σ is unbalanced but $\Sigma \setminus e$ is balanced.

Frustration: The *frustration index* $l(\Sigma)$ is the minimum number of edges whose deletion makes Σ balanced; that is, eliminates all negative circles. Similarly, the *frustration number* $l_0(\Sigma)$ is the minimum number of vertices whose deletion makes Σ balanced.

Suppose Σ is all negative, $\Sigma = (\Gamma, -)$. Then the negative circles are the odd circles and Σ is balanced if and only if Γ is bipartite.

Answers to the first six questions are known; but most of the problems are unsolved. If there is no "*Ans.*" line, the answer is unknown (at least to me). Each conjecture is an educated guess and may well be mistaken.

2 The Structure of the Class of Negative Circles

The solution to Problem 1 is essential to all work of this kind. The solution to Problem 2 is probably not needed at all, but it shows the difference in complexity between circles and chordless circles. Any question about circles can have the word "chordless" added to make a new question that is worth working on—but harder.

1. Can a given set $\mathcal{B}$ of circles in Γ be the negative circles of a signature?
   Ans. Easy: $\mathcal{B}$ is a negative circle set iff every theta subgraph of Γ has, among its three circles, either one or three that belong to $\mathcal{B}$ [18].
2. Can a given set of chordless circles in Γ be the negative chordless circles of a signature?